QUALIFYING - ALGEBRA January 1998

Choose 8 out of the following 10 problems. Write your code number and problem number on each sheet of paper.

1. Let G be a group and H a subgroup of finite index. Prove that H contains a subgroup K which is normal in G and of finite index in G.

2. Prove that the free group on 2 generators is not solvable. (Hint: One may use the fact that S_5 is generated by 2 elements.)

3. Give an example of an infinite group all of whose elements have finite order.

4. Let $\mathbf{Z}[\sqrt{10}]$ be the ring of all complex numbers of the form $a + b\sqrt{10}$ with $a, b \in \mathbf{Z}$. Prove that this ring is not a unique factorization domain.

5. Prove that if $f : A \to B$ and $g : B \to A$ are homomorphisms of $R-$modules (where R is a ring) such that $g \circ f = id_A$ then $B \simeq (Im\ f) \oplus (Ker\ g)$.

6. Prove that $(\mathbf{Z}/5\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q} = 0$. (Here $\mathbf{Z}$ is the ring of integers, $\mathbf{Q}$ is the field of rationals.)

7. Prove that the $\mathbf{Z}-$module $\mathbf{Q}$ of rationals is not projective.

8. Let A and B be two square matrices with complex coefficients such that $AB = BA$. Prove that A and B have a common eivenvector.

9. What is the Jordan normal form of the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

10. For each integer $n \geq 2$ let ζ_n is a primitive $n-$th root of unity in the field of complex numbers. Prove that the union

$$K = \bigcup_{n=2}^{\infty} \mathbf{Q}(\zeta_n)$$

is a field which is not algebraically closed.