

QUALIFYING EXAM - ALGEBRA – August 1998

Choose 8 out of the following 10 problems. Write your code number and problem number on each sheet of paper.

1. Let G be a group and H a subgroup of finite index. Prove that H contains a subgroup K which is normal in G and of finite index in G .
2. Prove that the free group on 2 generators is not solvable. (Hint: One may use the fact that S_5 is generated by 2 elements.)
3. Give an example of an infinite group all of whose elements have finite order.
4. Let $\mathbb{Z}[\sqrt{10}]$ be the ring of all complex numbers of the form $a + b\sqrt{10}$ with $a, b \in \mathbb{Z}$. Prove that this ring is not a unique factorization domain.
5. Prove that if $f : A \rightarrow B$ and $g : B \rightarrow A$ are homomorphisms of R -modules (where R is a ring) such that $g \circ f = id_A$ then $B \simeq (\text{Im } f) \oplus (\text{Ker } g)$.
6. Prove that $(\mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. (Here \mathbb{Z} is the ring of integers, \mathbb{Q} is the field of rationals.)
7. Prove that the \mathbb{Z} -module \mathbb{Q} of rationals is not projective.
8. Let A and B be two square matrices with complex coefficients such that $AB = BA$. Prove that A and B have a common eigenvector.
9. What is the Jordan normal form of the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

10. For each integer $n \geq 2$ let ζ_n be a primitive n -th root of unity in the field of complex numbers. Prove that the union

$$K = \bigcup_{n=2}^{\infty} \mathbb{Q}(\zeta_n)$$

is a field which is not algebraically closed.