Algebra Qualifying Examination
January 1999

Directions:
1. Solve all eight problems.
2. Start each questions on a new sheet of paper. Write only on one side of the page.

1. Let G be a finite group with $|G| = p^n$ where p is a prime

   (a) Prove that the center of G is nontrivial.

   (b) Prove that if $0 < m < n$ then G contains at least one normal subgroup of order p^m.

2. Show that if G is a finite group with $|G| = p^2 q$ with p and q distinct primes then G must contain a normal Sylow subgroup.

3. (a) Let ϕ and ψ be endomorphisms of a finite dimensional vector space over an algebraically closed field. Show that $\phi\psi = \psi\phi$ implies that ϕ and ψ have a common eigenvector.

   (b) Give a counterexample if the vectorspace is $\mathbb{R}^n$.

4. Let R be a commutative ring with unit and let S be a subring with unit. If $I \subset R$ is an ideal, prove

   (a) I maximal in $R \Rightarrow I \cap S$ maximal in S.

   (b) I prime in $R \Rightarrow I \cap S$ prime in S.

   (c) $I \cap S$ maximal in $S \not\Rightarrow I$ prime in S.

   (Give a counterexample in part (c)).

5. Let F be a field of characteristic 0 and let $\bar{F}$ be an algebraic closure of F. Let $f(x)$ be an irreducible polynomial in $F[X]$, let α be a root of f in $\bar{F}$ and let $L \subset \bar{F}$ be a splitting field of f. Prove that if the Galois group $\mathrm{Gal}(L/F)$ is abelian then $L = F(\alpha)$.

6. Let R be a commutative ring with identity. Show that the following are equivalent for an R-module P

   (a) Given

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle\alpha} \\ M & \xrightarrow{\varphi} & N \end{array}$$

   where M, N are R-modules, φ is a surjective R-module homomorphism and α is an R-module homomorphism then there is $\beta : P \to M$ (R-module homomorphism) with $\varphi \circ \beta = \alpha$.

   (b) Every exact sequence $O \to K \to M \to P \to O$ of R modules is split exact.

   (c) P is a direct summand of a free R-module.

7. Let K be a finite field. Let K^* be $K \backslash \{0\}$

   (a) Prove $|K| = p^n$ for some prime p.

   (b) Prove K^* is a cyclic group under multiplication.

8. Let F be a field with 5^{30} elements.

   (a) Describe the Galois group of F over $\mathbb{Z}_5$ (the field with five elements).

   (b) Find all subfields of F and relate them to the Galois group described in part (a).