Algebra Qualifying Exam

August 1999

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. Let p be a prime and let G be a group with order $|G| = p^n$. Prove that the center of G is non-trivial, i.e. prove that there is an element $z \in G$ with $z \neq e$ and such that $gz = zg$ for all $g \in G$.

2. Let R be a commutative ring with multiplicative identity 1. Show that R satisfies the ascending chain condition on ideals (i.e. whenever $I_1 \subset I_2 \subset \cdots$ is a nested sequence of ideals in R, there is an n such that $I_n = I_{n+1} = \cdots$) if and only if every ideal is finitely generated.

3. Prove that the polynomial ring $\mathbb{Z}[x]$ with integer coefficients is not a principal ideal domain.

4. Show that the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are similar over the rationals $\mathbb{Q}$.

5. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence of modules over a commutative ring R. Show that for any R-module D, the induced sequence

$$0 \longrightarrow \mathrm{Hom}(D, A) \xrightarrow{f_*} \mathrm{Hom}(D, B) \xrightarrow{g_*} \mathrm{Hom}(D, C)$$

is exact.

6. Show that every group G of order 56 has a nontrivial normal subgroup.

7. Let F_q denote the finite field with q elements, and let $f(x) \in F_q[x]$ be irreducible. Show that $f(x)$ divides $x^{q^n} - x$ if and only if the degree of f divides n.

8. Let ζ be a primitive nth root of unity in the complex number field $\mathbb{C}$. Show that $\mathbb{Q}(\zeta + \zeta^{-1})$ is Galois over $\mathbb{Q}$. Hint: First show by induction that if σ is an automorphism of $\mathbb{Q}(\zeta)$ leaving $\mathbb{Q}(\zeta + \zeta^{-1})$ fixed then for any integer k we have $\sigma(\zeta^k + \zeta^{-k}) = \zeta^k + \zeta^{-k}$.