Algebra Qualifying Exam

January 2000

Do any 8 of the following 10 problems. Show all your work and explain all steps in a proof or derivation. Indicate clearly which problems you are submitting.

1. Let G be a group whose automorphism group is cyclic. Show that G is abelian,

2. Prove that the symmetric group S_5 is not solvable.

3. Prove that the only groups of order 4 are $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and that these groups are not isomorphic.

4. Let R be a commutative ring with identity. Show that every maximal ideal of R is prime.

5. Find the Galois group of the polynomial $x^3 - 3$.

6. Prove that the multiplicative group consisting of all non-zero elements in a finite field is cyclic.

7. Let R be a ring and let $0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$ be a short exact sequence of R modules. Prove that if D is a projective R module, the Hom-functor $\mathrm{Hom}(D, \cdot)$ is exact. More explicitly show that for any R module D the sequence

$$0 \longrightarrow \mathrm{Hom}(D, A) \xrightarrow{\bar{\phi}} \mathrm{Hom}(D, B) \xrightarrow{\bar{\psi}} \mathrm{Hom}(D, C)$$

is exact and that if D is projective $\bar{\psi}$ is an epimorphism. Here $\bar{\phi}$ and $\bar{\psi}$ are the induced maps.

8. Give an example of a non-split short exact sequence of $\mathbb{Z}-$modules proving that your example does not split.

9. Show that any two commuting square matrices with complex coefficients have a common eigenvector.

10. Consider the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$T(x, y, z) = (2x, 2y, -x + 3y + 2z).$$

Show that 2 is the only eigenvalue of T, and find a basis for the eigenspace V_2 of T. Is T diagonalizable? Justify your answer.