MASTERS AND PH.D QUALIFYING EXAMINATION

ALGEBRA

August 2001

Instructions

Each of the following 10 problems is worth 10 points. Solve all of these 10 problems. Write your code number on each sheet of paper.

1) Suppose G is a group with no non-trivial, proper subgroups. Show that G is isomorphic to $\mathbf{Z}/p\mathbf{Z}$ for some prime p. (10 points)

2) Prove that the symmetric group S_3 is solvable. (10 points)

3) Prove that the multiplicative group of all non-zero rational numbers is not finitely generated. (10 points)

4) Let R be a commutative ring with multiplicative identity 1. Let I be the set of all elements $x \in R$ for which there exists an integer n, depending on x, such that $x^n = 0$. Show that I is an ideal in R. (10 points)

5) Prove that the ring $\mathbf{Z}[\sqrt{10}] = \{a + b\sqrt{10}; a, b \in \mathbf{Z}\}$ is not factorial. (10 points)

6) Consider the polynomial $x^4 - p \in \mathbf{Q}[x]$, where p is a prime natural number. Find a splitting field F of this polynomial over $\mathbf{Q}$ and show that the Galois group $Aut_{\mathbf{Q}}F$ in non-commutative. (10 points)

7) Suppose $K/\mathbf{Q}$ is a Galois extension of degree 3. Show that $K \subset \mathbf{R}$. (10 points)

8) Show that any algebraically closed field must be infinite. (10 points)

9) Let R be the ring of all rational numbers a/b such that b is not divisible by 17. Prove that R has only one maximal ideal.

10) Let ζ_{10} be a primitive tenth root of unity in $\mathbf{C}$. Show that $\mathbf{Q}(\zeta_{10})$ is a Galois extension of $\mathbf{Q}$ with Galois group $\mathbf{Z}/4\mathbf{Z}$. (10 points)