ALGEBRA QUALIFIER EXAM

There are 10 problems. Each problem is worth 10 points.

1. Show that the group of 2×2 invertible matrices with real coefficients is not solvable.

2. Let p be a prime. Show that there is an isomorphism of groups

$$\mathbf{Z}/p^2\mathbf{Z} \simeq (\mathbf{Z}/p\mathbf{Z}) \oplus (\mathbf{Z}/(p-1)\mathbf{Z}).$$

3. Let

$$M_1 = \frac{\mathbf{Q}[x]}{(x-2)^2}, \quad M_2 = \frac{\mathbf{Q}[x]}{(x-2)} \times \frac{\mathbf{Q}[x]}{(x-2)}.$$

1) Are M_1 and M_2 isomorphic as $\mathbf{Q}-$vector spaces ? Why ?
2) Are M_1 and M_2 isomorphic as rings ? Why ?
3) Are M_1 and M_2 isomorphic as $\mathbf{Q}[x]-$modules ? Why ?

4. Compute the center of the group of all isometries of the plane that send a regular pentagon into itself.

5. Prove that if A is an integral domain with quotient field K then K is not a free $A-$module unless A itself is a field.

6. Let

$$R = \{\text{continuous functions } f : [0,1] \to \mathbf{R}\},$$

$$I = \{f \in R : f(1/2) = 0\}.$$

Show that I is a maximal ideal in R.

7. Show that any element in a finite field is a sum of two squares.

8. Prove that the polynomial $t^5 - 7$ is irreducible in $\mathbf{Q}[t]$. Show that its Galois is solvable, non-commutative, of order 20.

9. Suppose M_1, M_2 are $n \times n$ matrices with complex coefficients such that $M_1 M_2 = M_2 M_1$. Show that M_1 and M_2 share a common eigenvector.

10. Show that there exists real (i.e. contained in $\mathbf{R}$) Galois extensions of $\mathbf{Q}$ of arbitrarily large degree.