JAN. 2004

Algebra Qualifying Exam

The total exam time is three hours. Each question is worth 20 points. Please put your answers to the questions on *separate* sheets of paper with your social security number (*not* your name) at the top of each page.

1. Viewing $\mathbf{Z}$ and $\mathbf{Q}$ as additive groups, show that $\mathbf{Q}/\mathbf{Z}$ is a torsion group. Moreover, for each integer $n \geq 1$ show that $\mathbf{Q}/\mathbf{Z}$ has one and only one subgroup of order n and that this subgroup is cyclic.

2. Let G be a cyclic group of order m and H a cyclic group of order n. If m and n are relatively prime show that $G \times H$ is cyclic of order mn.

3. A commutative ring A with identity is said to be Boolean if $x^2 = x$ for all $x \in A$.

a. Show that $2x = 0$ for all $x \in A$.

b. Show that every prime ideal $P \subset A$ is maximal and that A/P is a field with 2 elements.

4. Suppose A is a commutative ring with identity and let M and N be unitary A–modules. Suppose $f : M \rightarrow N$ and $g : N \rightarrow M$ are A–module homomorphisms satisfying $g(f(x)) = x$ for all $x \in M$. Show that there is an isomorphism

$$N = \mathrm{Im}(f) \oplus \mathrm{Ker}(g).$$

5. For each abelian group A of order 8, give an example of a Galois extension $K/\mathbf{Q}$ with $G_{K/\mathbf{Q}} \simeq A$.

6. Show that $\mathbf{Q}(\sqrt{-2})$ is not contained in any cyclic extension $K/\mathbf{Q}$ of degree 4.

7. Suppose R is an integral domain and k a field with $k \subset R$. If $\dim_k(R)$ is finite show that R is a field.