520 FINAL

There are 10 problems. Each problem is worth 10 points.

1. Let p and q be distinct primes. Prove that any group of order pq is solvable.

2. Let p be a prime. Prove that any group of order p^n has a non-trivial center.

3. Prove that the symmetric group S_4 is solvable.

4. Show that the group defined by generators a, b and relations $a^2 = b^3 = e$, $ab = b^2 a$ is isomorphic to the symmetric group S_3.

5. Prove that the additive group $\mathbf{Q}$ of rational numbers is not free abelian.

6. Prove that any maximal ideal in the ring R of real continuous functions on $[0, 1]$ is of the form $M_a = \{f \in R | f(a) = 0\}$ for some $a \in [0, 1]$.

7. Let K and L be fields. Prove that the product $K \times L$ in the category of rings cannot be a field.

8. Prove that if $\zeta \in \mathbf{C}$ is a primitive cubic root of unity then $\mathbf{Z}[\zeta]$ is Euclidean.

9. Prove that $\mathbf{Z}[\sqrt{10}]$ is not principal.

10. Let k be a field. Show that there exists a finitely generated module over the ring of polynomials $k[x, y]$ which is torsion free but not free. (Hint: consider an appropriate ideal.)

11. Give an example of an exact sequence of $\mathbf{Z}-$modules which is not split.

12. Prove that if $0 \to A \to B \to C \to 0$ is an exact sequence of $\mathbf{Z}-$modules with A and C finite sets of cardinality a and c with a and c coprime then the exact sequence is split.