Algebra Qualifying Exam

August 2005

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. Let p be a prime and let G be a group with order $|G| = p^n$. Prove that the center of G is non-trivial, i.e. prove that there is an element $z \in G$ with $z \neq e$ and such that $gz = zg$ for all $g \in G$.

2. Let R be a ring and $I \subset R$ an ideal. Suppose that $a \equiv b \bmod I$ and $c \equiv d \bmod I$.

(i) Show that $a + c \equiv (b + d) \bmod I$.

(ii) Show that $ac \equiv bd \bmod I$.

3. Show that the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are similar over the rationals $\mathbb{Q}$.

4. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence of modules over a commutative ring R. Show that if C is a free R module, then the exact sequence splits.

5. Let A, B be any two endomorphisms of a vector space V over $\mathbb{R}$, such that

$$A \circ B - B \circ A = Id,$$

where Id is the identity endomorphism. Show that V is infinite dimensional.

6. Show that no group of order 48 is simple.

7. Consider the ring $\mathbb{Z}[\sqrt{-5}]$.

(i) Find all the units in $\mathbb{Z}[\sqrt{-5}]$.

(ii) Show that $\mathbb{Z}[\sqrt{-5}]$ is an integral domain but not a Unique Factorization Domain (UFD).

8. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Q}$ the field of rational numbers. Prove that

$$(\mathbb{Z}/7\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$