Algebra Qualifying Exam

January 2006

Do the following 7 problems. Show all your work and explain all steps in a proof or derivation.

1. Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the 8-element group generated by quaternionic units with the usual quaternionic relations:

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k.$$

Let D_4 be the 8-element dihedral group generated by a, b with relations

$$a^4 = 1, \qquad a^k \neq 1 \quad \text{if } 0 < k < 4, \qquad b^2 = 1, \qquad ba = a^{-1}b.$$

Is D_4 isomorphic to Q? Prove your answer. (10 pts)

2. Consider the system of equations

$$x + y + z = 0, \qquad x + 3y + 4z = 0.$$

Show that the integer solutions of this system form a group isomorphic to $\mathbb{Z}$. (5 pts)

3. Determine all Abelian groups of order 36 up to isomorphism. (15 pts)

a) Give the decomposition of each group in terms of invariant factors $m_1, \cdots, m_t$ satisfying $m_1 | m_2 | \cdots | m_t$ as

$$G = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}.$$

b) Give the decomposition of each group in terms of elementary divisors $p_1^{s_1}, \cdots, p_r^{s_r}$ with p_i prime, as

$$G = \mathbb{Z}_{p_1^{s_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{s_r}}.$$

c) Give the isomorphism between the groups listed in a) with those listed in b).

4. Prove that any group of order 18 is solvable. (10 pts)

5. Let V be a real finite-dimensional vector space with a positive definite inner product $\langle \cdot, \cdot \rangle$. Let $L : V \to \mathbb{R}$ be a linear functional on V. Show that (10 pts)

$$\exists\, \vec{h} \in V \text{ such that } L(\vec{x}) = \langle \vec{x}, \vec{h} \rangle, \quad \vec{x} \in V.$$

6. Let R be a commutative ring with unity. Show that an element in R is nilpotent if and only if it belongs to every prime ideal of R. (10 pts)

7. Let E be a splitting field of the polynomial $x^5 - 2$ over the rationals $\mathbb{Q}$. Find the Galois group of $E/\mathbb{Q}$. (10 pts)