QUALIFYING EXAM - ALGEBRA 2007

Make sure to show all work and please solve each problem on a separate page. Please remember to put your social security number, rather than your name, on all pages of the exam.

1. Let G be a group and H a subgroup of finite index n.

- **a.** Prove that there is a homomorphism from G into the symmetric group on n letters S_n whose kernel is contained in H. Hint: think of the proof that a group of order n is isomorphic to a subgroup of S_n.
- **b.** Prove that there exists a normal subgroup N of G contained in H and of finite index in G.

2. Let G be a group whose group of automorphisms $\text{Aut}(G)$ is cyclic. Prove that G is abelian.

3. Let K and L be fields with $L = K(x)$, where x is transcendental over K. Let F be a subfield of L containing K. Assume that K is different from F.

- **a.** Prove that x is algebraic over F.
- **b.** Prove that L is a finite extension of F.

4. . Let $f : A \to B$ be a surjective homomorphism of commutative rings with 1 (in particular, we assume $f(1) = 1$). Prove that if A has a unique maximal ideal, then so does B.

5. Let $\mathcal{C}([0, 1])$ be the ring of continuous real valued functions on the closed interval $[0, 1]$.

- **a.** Suppose $S \subset [0, 1]$ is a subset and let $I(S) \subset \mathcal{C}(S)$ be the set of continuous functions which vanish at each point of S. Show that $I(S)$ is an ideal.
- **b.** When is $I(S)$ maximal?
- **c.** (Extra Credit!) Are all maximal ideals in $\mathcal{C}(S)$ of the type listed in part **b**?

6. Show that there are infinitely many distinct subfields $F \subset \mathbf{C}$ of the complex numbers with $[\mathbf{C} : F] = 2$.

7. Give an example of a vector space V and a linear map $L : V \to V$ which is injective but not surjective. Give an example of a vector space V and a linear map $M : V \to V$ which is surjective but not injective.

8. Show that $\sqrt[4]{2}$ is not contained in $\mathbf{Q}(\zeta_n)$ for any positive n: here ζ_n is a primitive n^{th} root of unity.

9. Characterize the maximal ideals of the ring $\mathbf{Z}[X]$.