QUALIFYING EXAM

August 2008

1. Prove that the group defined by generators a, b, c and the relation $abc = 1$ is infinite.

2. Give an example of an exact sequence of $\mathbf{Z}$-modules which is not split. Explain why your example works.

4. Prove that the ring $R := \mathbf{C}[x, y]/(y^2 - x^3)$ is an integral domain but not a unique factorization domain.

5. Prove that the ring $\mathbf{Q}(\sqrt{5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{7})$ is a field. Prove that the ring $\mathbf{Q}(\sqrt{5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{5})$ is not a field.

6. For each of the following groups, either give an injective homomorphism to $SL_2(\mathbf{C})$ or show that there is no injective homomorphism:
(a) $\mathbf{C}$
(b) $\mathbf{C}^*$
(c) $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$
(d) S_3

7. Find the minimal polynomial and the Jordan canonical form of the 3×3 complex matrix A assuming that $A^2 - 5A + 25I = 0$ and A is not a scalar matrix.

8. Suppose the cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ is irreducible over the rational numbers. Let K be its splitting field. When, in terms of the coefficients a, b, c, d, is the Galois group of $K/\mathbf{Q}$ isomorphic to S_3? When the Galois group is isomorphic to S_3, what is the unique quadratic extension $E/\mathbf{Q}$ contained in K? Justify your answer.

9. Suppose $K/\mathbf{Q}$ is a Galois extension with Galois group isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Is K isomorphic to $Q(\sqrt{a}, \sqrt{b}, \sqrt{c})$ for three (non-square) rational numbers a, b, c? Justify your answer.

10. Suppose $K \subset \mathbf{C}$ is a field. Suppose $\sqrt{2}$ is NOT contained in K but $\sqrt{2}$ is contained in every proper extension $\mathbf{C} \supset E \supset K$. Prove that E/K is a cyclic extension.