ALGEBRA QUALIFIER EXAM

There are 10 problems. Each problem is worth 10 points.

1. Let G be a subgroup of the additive group $\mathbf{R}$ of all real numbers and let ϵ be a positive real number. Assume G contains no positive element less than ϵ. Show that G is cyclic.

2. Give an example of a Galois extensions $K/\mathbf{Q}$ whose Galois group is $\mathbf{Z}/5\mathbf{Z}$.

3. Prove that the symmetric group S_4 is solvable.

4. Give an example of a ring with exactly two prime ideals.

5. Show that the free group on a set of 2 elements is not solvable.

6. Show that it is possible to embed **C** in **R** as an abelian group but not as a field.

7. Prove that the ideal $(2, 1+\sqrt{-5})$ in $\mathbf{Z}[\sqrt{-5}]$ is not principal.

8. Prove that the polynomial $t^3 + 2t + 1$ is irreducible in $\mathbf{Q}[t]$. Compute its Galois group.

9. Prove that if $\mathbf{Q}(t)$ is the field of rational functions in the variable t then the field extension $\mathbf{Q}(t^3+1) \subset \mathbf{Q}(t)$ is finite. Compute its degree.

10. Prove that there exist infinitely many field automorphisms $\phi : \mathbf{C} \to \mathbf{C}$ of order two.