ALGEBRA QUALIFYING EXAM - SUMMER 09

1. Prove that the free group on 2 generators is not isomorphic to the free group on 3 generators. Prove that $\mathbf{R}^2$ and $\mathbf{R}^3$ are isomorphic as groups. (Here $\mathbf{R}$ is the additive group of real numbers.)

2. Prove that the group of invertible upper triangular complex 2×2 matrices is solvable. Prove that the group of invertible complex 2×2 matrices is not solvable.

3. Prove that if p is a prime then any group with p^2 elements is abelian. Give an example of a non-abelian group with p^3 elements.

4. Prove that the ring $\mathbf{Z}[\sqrt{10}]$ is not a unique factorization domain.

5. Prove that $\mathbf{Q}$ is an injective $\mathbf{Z}$-module but not a projective $\mathbf{Z}$-module.

6. Suppose $C([0,1])$ is the ring of continuous functions on the closed interval [0,1]. Show that the set of maximal ideals of $C([0,1])$ is in bijection with the interval $[0,1]$.

7. For this problem, suppose $p \geq 3$ is a prime number and that $1 \leq a \leq p-1$.

a) Prove that the equation $x^2 \equiv a \pmod{p}$ has *either* two solutions *or* no solutions.

b) Prove that for *exactly half* of the choices of a there are two solutions and for the other half there are no solutions.

c) Show that the equation $x^3 \equiv a \pmod{p}$ for $1 \leq a \leq p-1$ either has no solutions, one solution, or three solutions. Moreover if there is one solution for a single value of a then there is one solution for *all* values of a.

8. Suppose $K/\mathbf{Q}$ is a Galois extension with $G_{K/\mathbf{Q}} \simeq S_3$. Is there necessarily a cubic polynomial $f(x) \in \mathbf{Q}[x]$ so that K is the splitting field of $f(x)$? Justify your answer.

9. Prove that if α and β are algebraic over $\mathbf{Q}$ of degrees m and n respectively, with m and n coprime, then the field $\mathbf{Q}(\alpha, \beta)$ has degree mn over $\mathbf{Q}$.

10. Let ζ_{17} denote a primitive 17^{th} root of unity and consider $\mathbf{Q}(\zeta_{17})/\mathbf{Q}$.

a) Show that $\mathbf{Q}(\zeta_{17})/\mathbf{Q}$ is Galois.

b) Find the Galois group $G_{\mathbf{Q}(\zeta_{17})/\mathbf{Q}}$, that is determine explicitly its structure and identify it as an abstract group (e.g. S_3, $\mathbf{Z}/3\mathbf{Z}$, etc.).

c) Find all subgroups of the Galois group.

d) How many quadratic extensions $K/\mathbf{Q}$ are contained in $\mathbf{Q}(\zeta_{17})$?