ALGEBRA

Qualifying exam August 2010

1. Let $f(X) \in \mathbf{Q}[X]$ be a polynomial of degree n. Let E be a splitting field of f over $\mathbf{Q}$. Show that $[E:\mathbf{Q}] \leq n!$.

2. Suppose K is a field of characteristic zero and G a finite group of automorphisms of K. Let K^G be the subfield of K fixed by G. Show that K/K^G is a Galois extention with Galois group G.

3. Suppose A, B are n by n matrix with complex coefficients. Show that $AB - BA$ cannot be equal to the identity matrix.

4. Consider the derivative map $D : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$ given by $D(f(t)) = f'(t)$ and the multiplication by t map $M : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$ given by $M(f(t)) = tf(t)$. (Here $C^\infty(\mathbf{R})$ denotes the vector space of C^∞ functions $\mathbf{R} \to \mathbf{R}$.) Compute the eigenvalues (and the corresponding eigenvectors) of the maps D, M, and $D \circ M - M \circ D$.

5. Let $M = \mathbf{C}(z)$ be the field of rational functions of z with $\mathbf{C}$ coefficients. Show that the map $SL_2(\mathbf{C}) \to Aut_{\mathbf{C}}(M)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \sigma, \quad \sigma(f(z)) = f\left(\frac{az+b}{cz+d}\right)$$

is a group homomorphism. Compute the kernel and the image of this homomorphism.

6. Compute the center of the symmetric group S_n, $n \geq 3$.

7. Prove that the group defined by generators a, b and one relation $a^2b^3 = e$ is infinite.

8. Prove that if p is an odd prime number then the group of invertible elements in the ring $\mathbf{Z}/p^n\mathbf{Z}$ is cyclic.

9. Prove that the ring $\mathbf{Z}[\sqrt{-5}]$ is not principal.

10. Prove that the ring $\{\frac{n}{m}; n, m \in \mathbf{Z}, m \notin 5\mathbf{Z}\}$ is local.