# ALGEBRA QUALIFIER EXAMINATION

#### JANUARY 2013

There are 10 problems. Each problem is worth 10 points. Please write your banner ID on the top of this page.

- (1) Let p be a prime. Show that any group with  $p^2$  elements is abelian. Give an example of a non-abelian group with  $p^3$  elements.
- (2) Suppose that N is a normal subgroup of a group G, with  $|N| < \infty$ . Suppose also that H is a subgroup of G with  $[G:H] < \infty$  and |N| and [G:H] relatively prime. Show that  $N \subseteq H$ .
- (3) Show there are no simple groups of order 196.
- (4) Let F be the free group on two generators and [F, F] its commutator. Prove that F/[F, F] is isomorphic to  $\mathbb{Z} \times \mathbb{Z}$ .
- (5) Let R be a commutative ring and let I and J be ideals such that I+J=R. Prove that there is an isomorphism of R-modules  $I \oplus J \simeq R \oplus IJ$ . Conclude that if I and J are projective R-modules, so is IJ.
- (6) Let  $I = (x^5, y^4 z^6)$  be an ideal of the polynomial ring k[x, y, z]. Determine the nilradical of k[x, y, z]/I.
- (7) (a) Show that  $\mathbb{Z}/a\mathbb{Z} \otimes \mathbb{Z}/b\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$  where d = (a, b).
  - (b) Then use part (a) to show for any finite abelian group A of order n with  $p^k$  the largest power of the prime p dividing n that  $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$  is isomorphic to the Sylow p-subgroup of A.
- (8) Determine the minimal polynomial for  $\alpha = 1 + \sqrt[4]{2}$  over  $\mathbb{Q}$ . Then find the splitting field of  $\mathbb{Q}(\alpha)$ .
- (9) Let p and q be primes. Find the Galois group of the polynomial  $x^p q$  over  $\mathbb{Q}$ .
- (10) Let k(x) be the field of rational functions in one variable over a field k. Prove that the degree of the extension  $k(\frac{x^5}{x^2+1}) \subset k(x)$  is 5.

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# ALGEBRA QUALIFIER EXAMINATION

#### JANUARY 2012

There are 10 problems. Each problem is worth 10 points. Please write your name on the top of this page.

- (1) How many elements of order 7 must be in a simple group of order 168?
- (2) Prove that any group with pq elements (p and q unequal primes) is solvable.
- (3) Prove that the group with generators a, b and relations  $a^3 = b^2 = e$ ,  $ab = ba^2$  is isomorphic to the symmetric group  $S_3$ .
- (4) Prove that  $\mathbb{Z}[\sqrt{10}]$  is not a unique factorization domain.
- (5) Let F be a field and  $R = F[x, x^2y, x^3y^2, x^4y^3, \ldots]$  be a subring of the polynomial ring F[x, y]. Prove that the field of fractions of R is the same as F(x, y), the field of fractions of F[x, y].
- (6) Prove that  $\mathbb{Q}$  is not a projective  $\mathbb{Z}$ -module.
- (7) Let I = (2, x) be the ideal generated by 2 and x in  $\mathbb{Z}[x]$ .
  - (a) Show that  $2 \otimes x x \otimes 2$  is a torsion element of  $I \otimes_R I$ .
  - (b) Show that the submodule of  $I \otimes_R I$  generated by  $2 \otimes x x \otimes 2$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ .
- (8) Let y and z be indeterminates and  $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$ . Suppose  $u = y^{20}$  and  $v = z^{50}$ . Determine the separable closure of  $\mathbb{Z}_5(u,v)$  in  $\mathbb{Z}_5(y,z)$
- (9) Find the Galois group of  $f(x) = x^4 2x^2 4$  over  $\mathbb{Q}$ .
- (10) Prove that if an irreducible polynomial  $f \in \mathbb{Q}[x]$  has prime degree p and has exactly p-2 real roots in  $\mathbb{C}$  then the Galois group of f over  $\mathbb{Q}$  is isomorphic to the symmetric group  $S_p$ .

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# ALGEBRA QUALIFIER EXAMINATION

#### AUGUST 2012

There are 10 problems. Each problem is worth 10 points. Please write your banner ID on the top of this page.

- (1) Let p, q and r be primes. Show that no group of order pqr is simple.
- (2) Prove that the group defined by generators a, b, c, d and relations  $adb = b^2 a$ ,  $a^3 = c$ ,  $b^2 = c$ ,  $d^2 = c$  is infinite and non-commutative.
- (3) Let  $\sigma$  and  $\tau$  be distinct 3-cycles in  $S_5$ . Show that  $\langle \sigma, \tau \rangle$  is either isomorphic to  $A_4$  or is  $A_5$ .
- (4) Let R be a domain. Prove that if R contains a non-principal proper ideal then there is an ideal J maximal with this property. Prove that any such J is a prime ideal. Conclude that if all prime ideals of R are principal then R is a principal ideal domain.
- (5) Prove that a finite abelian group is neither projective nor injective.
- (6) Let  $L = \mathbb{Q}(\sqrt[8]{2}, i)$  and  $E_1 = \mathbb{Q}(i)$ ,  $E_2 = \mathbb{Q}(\sqrt{2})$  and  $E_3 = \mathbb{Q}(i\sqrt{2})$  Find the Galois groups:  $G(L/E_1)$ ,  $G(L/E_2)$  and  $G(L/E_3)$ .
- (7) Prove that if a rational function  $f(x) \in \mathbb{Q}(x)$  in one variable x has the property that  $f(x) = f(-\frac{1}{x})$  then there exists a rational function  $g \in \mathbb{Q}(x)$  such that  $f(x) = g(\frac{x^2-1}{x})$ .
- (8) Prove that the exact sequence  $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \to 0$  with  $\alpha(x) = 2x$ ,  $\beta(x) = x + 2\mathbb{Z}$  is exact and non-split.
- (9) Prove that if R is an integral domain, I is a non-zero ideal, and F is the quotient field of R then  $F \otimes_R (R/I) = 0$ .
- (10) Prove that if M is a free  $\mathbb{Z}$ -module of infinite rank then the  $\mathbb{Z}$ -module  $Hom_{\mathbb{Z}}(M,\mathbb{Z})$  is not isomorphic to M.

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# University of New Mexico Department of Mathematics and Statistics Algebra Qualifying Exam January 2011

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score. When solving a problem with multiple parts you can assume the validity of all previous parts even if you have not solved them.

- (1) Let  $f(x) = x^3 2x 2 \in \mathbb{Q}[x]$ .
  - (a) Show that f(x) is irreducible.
  - (b) If  $\theta$  is a complex root of f(x), express  $\theta^{-1}$  as a polynomial in  $\theta$  with rational coefficients.
- (2) Prove that  $f(x)^p = f(x^p)$  for any polynomial in  $\mathbb{Z}_p[x]$ , where  $\mathbb{Z}_p$  is the finite field with p elements.
- (3) Let  $\alpha : F(X) \to F(X)$  be a field homomorphism satisfying  $\alpha(X) = f(X)$ . Let L be the image of  $\alpha$ . Show that F(X)/L is a finite extension and find the minimal polynomial of X over L.
- (4) Let G be the subgroup of  $GL_2(\mathbb{R})$  generated by  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . Find all automorphisms of G.
- (5) If P is a Sylow p-subgroup of G and N is a subgroup of G, show that  $P \cap N$  is a Sylow p-subgroup of N.
- (6) Let  $A_n$  denote the alternating group on n elements. Show that any group of finite order is isomorphic to a subgroup of  $A_n$  for some n.
- (7) Let G be the subgroup of  $GL_2(\mathbb{R})$  of all matrices of the form

$$\left[\begin{array}{cc}a&b\\0&1\end{array}\right]$$

where  $a \neq 0$ .

- (a) Find the commutator subgroup of G.
- (b) Is G solvable?
- (8) Give an example of a projective module over  $\mathbb{Z}_{20}$  which is not free. Justify your answer.
- (9) Suppose  $C_0(0, 1]$  is the ring of continuous real-valued functions on the closed interval [0, 1] that vanish at zero. Show that the set of maximal ideals of  $C_0(0, 1]$  is uncountable.
- (10) Suppose R is a commutative ring with unity.
  - (a) Show that if a is nilpotent, then 1 + a is invertible.
  - (b) Show that if a and b are nilpotent and r is any element of R, ra is nilpotent and a + b are nilpotent.
  - (c) Show that if  $a_0$  is a unit and  $a_i$  are nilpotent for  $1 \le i \le n$  then  $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$  is invertible.

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# ALGEBRA Qualifying exam August 2010

- 1. Let  $f(X) \in \mathbf{Q}[X]$  be a polynomial of degree n. Let E be a splitting field of f over  $\mathbf{Q}$ . Show that  $[E:\mathbf{Q}] \leq n!$ .
- **2.** Suppose K is a field of characteristic zero and G a finite group of automorphisms of K. Let  $K^G$  be the subfield of K fixed by G. Show that  $K/K^G$  is a Galois extention with Galois group G.
- **3.** Suppose A, B are n by n matrix with complex coefficients. Show that AB BA cannot be equal to the identity matrix.
- 4. Consider the derivative map  $D: C^{\infty}(\mathbf{R}) \to C^{\infty}(\mathbf{R})$  given by D(f(t)) = f'(t) and the multiplication by t map  $M: C^{\infty}(\mathbf{R}) \to C^{\infty}(\mathbf{R})$  given by M(f(t)) = tf(t). (Here  $C^{\infty}(\mathbf{R})$  denotes the vector space of  $C^{\infty}$  functions  $\mathbf{R} \to \mathbf{R}$ .) Compute the eigenvalues (and the corresponding eigenvectors) of the maps D, M, and  $D \circ M M \circ D$ .
- **5.** Let  $M = \mathbf{C}(z)$  be the field of rational functions of z with  $\mathbf{C}$  coefficients. Show that the map  $SL_2(\mathbf{C}) \to Aut_{\mathbf{C}}(M)$  given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \sigma, \quad \sigma(f(z)) = f\left(\frac{az+b}{cz+d}\right)$$

is a group homomorphism. Compute the kernel and the image of this homomorphism.

- **6.** Compute the center of the symmetric group  $S_n$ ,  $n \geq 3$ .
- 7. Prove that the group defined by generators a, b and one relation  $a^2b^3 = e$  is infinite.
- 8. Prove that if p is an odd prime number then the group of invertible elements in the ring  $\mathbf{Z}/p^n\mathbf{Z}$  is cyclic.
- **9.** Prove that the ring  $\mathbf{Z}[\sqrt{-5}]$  is not principal.
- 10. Prove that the ring  $\{\frac{n}{m}; n, m \in \mathbb{Z}, m \notin 5\mathbb{Z}\}$  is local.

Original Fall 2010