ALGEBRA QUALIFIER EXAMINATION

JANUARY 2013

There are 10 problems. Each problem is worth 10 points. Please write your banner ID on the top of this page.

(1) Let p be a prime. Show that any group with p^2 elements is abelian. Give an example of a non-abelian group with p^3 elements.

(2) Suppose that N is a normal subgroup of a group G, with $|N| < \infty$. Suppose also that H is a subgroup of G with $[G : H] < \infty$ and $|N|$ and $[G : H]$ relatively prime. Show that $N \subseteq H$.

(3) Show there are no simple groups of order 196.

(4) Let F be the free group on two generators and $[F, F]$ its commutator. Prove that $F/[F, F]$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

(5) Let R be a commutative ring and let I and J be ideals such that $I + J = R$. Prove that there is an isomorphism of R-modules $I \oplus J \simeq R \oplus IJ$. Conclude that if I and J are projective R-modules, so is IJ.

(6) Let $I = (x^5, y^4z^6)$ be an ideal of the polynomial ring $k[x, y, z]$. Determine the nilradical of $k[x, y, z]/I$.

(7) (a) Show that $\mathbb{Z}/a\mathbb{Z} \otimes \mathbb{Z}/b\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$ where $d = (a, b)$.
(b) Then use part (a) to show for any finite abelian group A of order n with p^k the largest power of the prime p dividing n that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.

(8) Determine the minimal polynomial for $\alpha = 1 + \sqrt[4]{2}$ over $\mathbb{Q}$. Then find the splitting field of $\mathbb{Q}(\alpha)$.

(9) Let p and q be primes. Find the Galois group of the polynomial $x^p - q$ over $\mathbb{Q}$.

(10) Let $k(x)$ be the field of rational functions in one variable over a field k. Prove that the degree of the extension $k(\frac{x^5}{x^2+1}) \subset k(x)$ is 5.

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JANUARY 2012

There are 10 problems. Each problem is worth 10 points. Please write your name on the top of this page.

(1) How many elements of order 7 must be in a simple group of order 168?

(2) Prove that any group with pq elements (p and q unequal primes) is solvable.

(3) Prove that the group with generators a, b and relations $a^3 = b^2 = e,\ ab = ba^2$ is isomorphic to the symmetric group S_3.

(4) Prove that $\mathbb{Z}[\sqrt{10}]$ is not a unique factorization domain.

(5) Let F be a field and $R = F[x, x^2y, x^3y^2, x^4y^3, \ldots]$ be a subring of the polynomial ring $F[x, y]$. Prove that the field of fractions of R is the same as $F(x, y)$, the field of fractions of $F[x, y]$.

(6) Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

(7) Let $I = (2, x)$ be the ideal generated by 2 and x in $\mathbb{Z}[x]$.
  (a) Show that $2 \otimes x - x \otimes 2$ is a torsion element of $I \otimes_R I$.
  (b) Show that the submodule of $I \otimes_R I$ generated by $2 \otimes x - x \otimes 2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(8) Let y and z be indeterminates and $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$. Suppose $u = y^{20}$ and $v = z^{50}$. Determine the separable closure of $\mathbb{Z}_5(u, v)$ in $\mathbb{Z}_5(y, z)$

(9) Find the Galois group of $f(x) = x^4 - 2x^2 - 4$ over $\mathbb{Q}$.

(10) Prove that if an irreducible polynomial $f \in \mathbb{Q}[x]$ has prime degree p and has exactly $p-2$ real roots in $\mathbb{C}$ then the Galois group of f over $\mathbb{Q}$ is isomorphic to the symmetric group S_p.

ALGEBRA QUALIFIER EXAMINATION

AUGUST 2012

There are 10 problems. Each problem is worth 10 points. Please write your banner ID on the top of this page.

(1) Let p, q and r be primes. Show that no group of order pqr is simple.

(2) Prove that the group defined by generators a, b, c, d and relations $adb = b^2a$, $a^3 = c$, $b^2 = c$, $d^2 = c$ is infinite and non-commutative.

(3) Let σ and τ be distinct 3-cycles in S_5. Show that $< \sigma, \tau >$ is either isomorphic to A_4 or is A_5.

(4) Let R be a domain. Prove that if R contains a non-principal proper ideal then there is an ideal J maximal with this property. Prove that any such J is a prime ideal. Conclude that if all prime ideals of R are principal then R is a principal ideal domain.

(5) Prove that a finite abelian group is neither projective nor injective.

(6) Let $L = \mathbb{Q}(\sqrt[8]{2}, i)$ and $E_1 = \mathbb{Q}(i)$, $E_2 = \mathbb{Q}(\sqrt{2})$ and $E_3 = \mathbb{Q}(i\sqrt{2})$ Find the Galois groups: $G(L/E_1)$, $G(L/E_2)$ and $G(L/E_3)$.

(7) Prove that if a rational function $f(x) \in \mathbb{Q}(x)$ in one variable x has the property that $f(x) = f(-\frac{1}{x})$ then there exists a rational function $g \in \mathbb{Q}(x)$ such that $f(x) = g(\frac{x^2-1}{x})$.

(8) Prove that the exact sequence $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \to 0$ with $\alpha(x) = 2x$, $\beta(x) = x + 2\mathbb{Z}$ is exact and non-split.

(9) Prove that if R is an integral domain, I is a non-zero ideal, and F is the quotient field of R then $F \otimes_R (R/I) = 0$.

(10) Prove that if M is a free $\mathbb{Z}$-module of infinite rank then the $\mathbb{Z}$-module $Hom_{\mathbb{Z}}(M, \mathbb{Z})$ is not isomorphic to M.

University of New Mexico
Department of Mathematics and Statistics
Algebra Qualifying Exam
January 2011

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score. When solving a problem with multiple parts you can assume the validity of all previous parts even if you have not solved them.

(1) Let $f(x) = x^3 - 2x - 2 \in \mathbb{Q}[x]$.
  (a) Show that $f(x)$ is irreducible.
  (b) If θ is a complex root of $f(x)$, express θ^{-1} as a polynomial in θ with rational coefficients.

(2) Prove that $f(x)^p = f(x^p)$ for any polynomial in $\mathbb{Z}_p[x]$, where $\mathbb{Z}_p$ is the finite field with p elements.

(3) Let $\alpha : F(X) \to F(X)$ be a field homomorphism satisfying $\alpha(X) = f(X)$. Let L be the image of α. Show that $F(X)/L$ is a finite extension and find the minimal polynomial of X over L.

(4) Let G be the subgroup of $GL_2(\mathbb{R})$ generated by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find all automorphisms of G.

(5) If P is a Sylow p-subgroup of G and N is a subgroup of G, show that $P \cap N$ is a Sylow p-subgroup of N.

(6) Let A_n denote the alternating group on n elements. Show that any group of finite order is isomorphic to a subgroup of A_n for some n.

(7) Let G be the subgroup of $GL_2(\mathbb{R})$ of all matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

where $a \neq 0$.
  (a) Find the commutator subgroup of G.
  (b) Is G solvable?

(8) Give an example of a projective module over $\mathbb{Z}_{20}$ which is not free. Justify your answer.

(9) Suppose $C_0(0, 1]$ is the ring of continuous real-valued functions on the closed interval $[0, 1]$ that vanish at zero. Show that the set of maximal ideals of $C_0(0, 1]$ is uncountable.

(10) Suppose R is a commutative ring with unity.
  (a) Show that if a is nilpotent, then $1 + a$ is invertible.
  (b) Show that if a and b are nilpotent and r is any element of R, ra is nilpotent and $a + b$ are nilpotent.
  (c) Show that if a_0 is a unit and a_i are nilpotent for $1 \leq i \leq n$ then $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ is invertible.

University of New Mexico
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Algebra Qualifying Exam
January 2011

Instructions: Complete all problems to get full credit. [illegible] number the pages, and put [illegible] on each page. Clear and concise answers [illegible]. When applying a result in [illegible] state the result precisely [illegible] even if you have not [illegible] it.

(1) Let $f(x) = x^3 - 2x - 2 \in \mathbb{Q}[x]$.
(a) Show that $f(x)$ is irreducible.
(b) If θ is a complex root of $f(x)$, [illegible] rational numbers.

(2) Prove that [illegible] field with p elements.

(3) Let [illegible] $F(X)$ [illegible]. Show that $F(X)$ [illegible] and find the minimal polynomial of X over [illegible].

(4) Let G be the subgroup of $GL_2(\mathbb{R})$ generated by [illegible]. Find all automorphisms of G.

(5) If P is a Sylow p-subgroup of G and N is a normal subgroup of G, show that $P \cap N$ is a Sylow p-subgroup of N.

(6) Let A_n denote the alternating group on n elements. Show that any group of odd order is isomorphic to a subgroup of A_n for some n.

(7) Let G be the subgroup of $GL_2(\mathbb{R})$ of all matrices of the form

$$\begin{bmatrix} [illegible] & b \\ [illegible] & [illegible] \end{bmatrix}$$

where $ac \neq 0$.
(a) Find the commutator subgroup of G.
(b) Is G solvable?

(8) Give an example of a projective module over $\mathbb{Z}_6$ which is not free. Justify your answer.

(9) Suppose $C[0,1]$ is the ring of [illegible] continuous [illegible] functions on [illegible] $[0,1]$ and vanish at zero. Show that [illegible] of $C[0,1]$ is uncountable.

(10) Suppose R is a commutative ring with 1.
(a) Show that if a is nilpotent then $1 + a$ is invertible.
(b) Show that if a and b are nilpotent [illegible] and $a + b$ are nilpotent.
(c) Show that if a_0 is a unit and a_i are nilpotent for $1 \le i \le n$ then $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$ is invertible.

algebra (4)

ALGEBRA

Qualifying exam August 2010

1. Let $f(X) \in \mathbf{Q}[X]$ be a polynomial of degree n. Let E be a splitting field of f over $\mathbf{Q}$. Show that $[E : \mathbf{Q}] \leq n!$.

2. Suppose K is a field of characteristic zero and G a finite group of automorphisms of K. Let K^G be the subfield of K fixed by G. Show that K/K^G is a Galois extention with Galois group G.

3. Suppose A, B are n by n matrix with complex coefficients. Show that $AB - BA$ cannot be equal to the identity matrix.

4. Consider the derivative map $D : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$ given by $D(f(t)) = f'(t)$ and the multiplication by t map $M : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$ given by $M(f(t)) = tf(t)$. (Here $C^\infty(\mathbf{R})$ denotes the vector space of C^∞ functions $\mathbf{R} \to \mathbf{R}$.) Compute the eigenvalues (and the corresponding eigenvectors) of the maps D, M, and $D \circ M - M \circ D$.

5. Let $M = \mathbf{C}(z)$ be the field of rational functions of z with $\mathbf{C}$ coefficients. Show that the map $SL_2(\mathbf{C}) \to Aut_{\mathbf{C}}(M)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \sigma, \quad \sigma(f(z)) = f\left(\frac{az+b}{cz+d}\right)$$

is a group homomorphism. Compute the kernel and the image of this homomorphism.

6. Compute the center of the symmetric group S_n, $n \geq 3$.

7. Prove that the group defined by generators a, b and one relation $a^2b^3 = e$ is infinite.

8. Prove that if p is an odd prime number then the group of invertible elements in the ring $\mathbf{Z}/p^n\mathbf{Z}$ is cyclic.

9. Prove that the ring $\mathbf{Z}[\sqrt{-5}]$ is not principal.

10. Prove that the ring $\{\frac{n}{m}; n, m \in \mathbf{Z}, m \notin 5\mathbf{Z}\}$ is local.

Original
Fall 2010