ALGEBRA QUALIFYING EXAMINATION

There are 10 problems. Each problem is worth 10 points. Please write your banner ID on the top of each page.

(1) Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = \{e\}$.

(2) Let G be a group and H be a subgroup contained in the center of G. Show that if G/H is cyclic then G is abelian.

(3) Show no group of order 132 is simple.

(4) Suppose $I \subset \mathbb{Z}$ is a proper ideal. Define the localization $\mathbb{Z}_I$ of $\mathbb{Z}$ at I. If I and J are both ideals in $\mathbb{Z}$ when is $\mathbb{Z}_I \simeq \mathbb{Z}_J$?

(5) Suppose $R = \mathbb{C}[X_1, \dots, X_n]$ and suppose $M \subset R$ is a maximal ideal. Show that there are complex numbers $a_1, \dots, a_n$ so that M is the ideal generated by

$$X_1 - a_1, \dots, X_n - a_n.$$

(6) Show that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, but is not a flat $\mathbb{Z}$-module.

(7) Let V be an n-dimensional vector space over F. Show that the dual vector space $V^* = \mathrm{Hom}_F(V, F)$ is also n-dimensional.

(8) Let $\mathbb{F}_p$ denote the field with p elements. Suppose $\mathbb{F}$ is a finite field of characteristic p. Show that $\mathbb{F}$ is a cyclic extension of $\mathbb{F}_p$.

(9) Let $K = \mathbb{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}]$. Prove that K is a Galois extension of $\mathbb{Q}$. Find all intermediate field extensions and prove that your list is complete.

(10) Construct a Galois extension L of $\mathbb{Q}$ whose Galois group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

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