ALGEBRA QUALIFYING EXAMINATION

There are 10 problems. Each problem is worth 10 points. Please write your banner ID on the top of each page.

(1) Let G be a group. Prove that G is abelian if and only if the map $f : G \to G$ defined by $f(g) = g^2$ is a homomorphism.

(2) Show that the diagonal subgroup $D = \{(a, a) \mid a \in S_3\}$ in $S_3 \times S_3$ is not normal.

(3) Let G be a group of order 105 with a normal Sylow 3 subgroup. Show that G is abelian.

(4) Suppose that

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' \end{array}$$

is a commutative diagram of abelian groups with exact rows. Prove that if α is surjective and β and δ are injective, then γ is injective.

(5) Let R be a commutative ring and I and J ideals of R. Show that

$$R/I \otimes_R R/J \cong R/(I + J).$$

(6) Give an example of a vector space V and a linear map $L : V \to V$ which is injective but not surjective. Give an example of a linear map $L' : V \to V$ which is surjective but not injective.

(7) Determine how many elements the following rings have and determine whether or not they are fields:
  a: $\mathbb{Z}[X]/(3, X^2 + 2)$.
  b: $\mathbb{Z}[X]/(5, X^3 + X + 1)$.

(8) Suppose α and $\frac{1}{\alpha}$ are roots of a cubic polynomial $f(X) \in \mathbb{Z}[X]$. What could the Galois group of f be? Give an example for each possibile Galois group.

(9) State and prove the reduction criterion modulo a prime p for irreducibility of a polynomial $f(X) \in \mathbb{Z}[X]$.

(10) Suppose $K \subset \mathbb{C}$ is a *maximal* subfield with the property that $\sqrt{2} \notin K$. If F/K is a finite field extension show that F is a cyclic extension of K.

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