ALGEBRA QUALIFYING EXAMINATION: AUGUST 2014

There are 10 problems. Each problem is worth 10 points. Please write your banner ID on the top of each page.

(1) A finite group G acts on itself by conjugation. Determine all possible G if this action yields precisely three distinct orbits.

(2) Let G be the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a \in (\mathbb{Z}/p\mathbb{Z})^{\times}$ and $b \in \mathbb{Z}/p\mathbb{Z}$. Find all normal subgroups of G.

(3) Let R be a commutative ring and $r \in R$ a non-nilpotent element. (i.e. There exists no natural number n with $r^n = 0$.)

(a) Show that there is a prime ideal P such that $r \notin P$.

(b) Give an example of a ring R and a non-nilpotent element r which is contained in all maximal ideals $\mathfrak{m}$ of R.

(4) Let η be a primitive 16th root of unity over a field F. What can you say about the degree $[F(\eta) : F]$ if:

(a) F has 17 elements.

(b) F has 9 elements.

(c) The characteristic of F is 2.

(5) Let k be a field. Show that $\{x^i \otimes y^j \mid i, j \geq 0\}$ is a basis for $k[x] \otimes_k k[y]$ as a k-vector space. Then show that $k[x] \otimes_k k[y] \cong k[x, y]$ as k-vector spaces.

(6) Let p be a prime integer and let $f(x)$ be the polynomial $x^p - p - 1$.

(a) Prove that $f(x)$ irreducible in $\mathbb{Q}[x]$.

(b) Find the Galois group of $f(x)$ over $\mathbb{Q}$.

(7) Prove that the field $\mathbb{R}$, the real numbers, has no automorphisms besides the identity.

(8) Let R be the ring of all continuous real valued functions defined on the closed interval $[0, 1]$ and let M be a maximal ideal in R. Prove that there exists $\alpha \in [0, 1]$ such that $M = \{f \in R;\ f(\alpha) = 0\}$.

(9) Let $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \to 0$ be the sequence defined by $\alpha(n) = 2n$, $\beta(m) = m + 2\mathbb{Z}$.

(a) Prove that the sequence is exact.

(b) Prove that the sequence is not split.

(10) Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module. Is this module injective?