ALGEBRA QUALIFIER JANUARY 2017

Instructions: Please complete the following 10 problems working only on one side of each sheet of paper.

(1) Suppose p and q are primes such that $p \nmid q \pm 1$ and $q \nmid p \pm 1$ and G is a group of order p^2q^2. Show that G is abelian.

(2) Determine the number of p-Sylow subgroups in S_p.

(3) Let $K \subseteq H$ be a chain of subgroups in G.
   (a) If H is characteristic in G and K is characteristic in H. Show that K is characteristic in G.
   (b) Show that if in the above statement characteristic is replaced by normal in all its occurences then the amended statement is false.

(4) Let I and J be two ideals in a commutative ring R such that $I + J = R$. Prove that:
   (a) $IJ = I \cap J$.
   (b) $I^n + J^n = R$ for all n.

(5) Let k be any field. Show that $k[t^2, t^3]$ is not a unique factorization domain.

(6) Determine $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$.

(7) Show that the center of a division ring is a field.

(8) Determine the splitting field K of $x^6 + 8$ over $\mathbb{Q}$.

(9) Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and let G be the group of field automorphisms of $\overline{\mathbb{Q}}$. Show that G is uncountable and non-abelian.

(10) Prove that the polynomial $x^n + y^n + z^n$ is irreducible in $\mathbb{C}[x, y, z]$ for all $n \geq 2$.