ALGEBRA QUALIFIER AUGUST 2017

Instructions: There are 10 problems below. Start each problem on a separate sheet, each labeled with your student code and the problem number. For uniformity, please only use one side of the paper when writing your solutions. If you use more than one sheet for a problem, please label accordingly.

(1) Suppose N and H are normal subgroups of G such that G/N and G/H are abelian. Show that $G/(H \cap N)$ is abelian.

(2) Show that any group of order 100 is not simple.

(3) Suppose p and q are distinct primes. Show that if $p \nmid q-1$ and $q \nmid p \pm 1$, then a group of order p^2q must be abelian.

(4) Suppose R and S are rings and $I \subseteq R$ is an ideal and $J \subseteq S$ is an ideal. Show that $I \times J$ is an ideal of $R \times S$. Moreover, show that every ideal of $R \times S$ is of this form.

(5) Let A be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that A is not an integral domain. Does A contain idempotent elements different from 0 and 1? Does A contain nilpotent elements other than 0? Justify your answers.

(6) Show that $k[x^2, x^3]$ is not a unique factorization domain.

(7) Determine all the $\mathbb{Z}_{84}$-projective submodules of $\mathbb{Z}_{84}$. Are they also injective? Give reasons for your answer.

(8) Show that $x^5 - 1 = (x-1)(x^2 - 4x + 1)(x^2 + 5x + 1)$ is a factorization of $x^5 - 1$ into irreducibles in $\mathbb{F}_{19}[x]$. Give a single representative for each similarity class of 2×2 matrices A with coefficients in $\mathbb{F}_{19}$ which satisfy $A^5 = I$.

(9) What is the splitting field K of $x^6 + 3$ over $\mathbb{Q}$? Find the Galois group of K over $\mathbb{Q}$.

(10) Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and let G be the group of automorphisms of the field $\overline{\mathbb{Q}}$. Prove that G is uncountable.