Qualifying Exam – Algebra
January 2018

This exam consists of 10 problems. To make sure all of your solutions are considered, please write your answers only on one side of each sheet of paper you turn in. Enter your student code at the top right of each sheet.

1. Show that $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ is a Galois extension. What is the unique quadratic sub-extension?

2. Is there a degree 4 extension $F/\mathbb{Q}$ which contains no quadratic subextension? Prove that your answer is correct.

3. Let K be the splitting field of $x^5 - 7$ over $\mathbb{Q}$. Prove that the Galois group $G_{K/\mathbb{Q}}$ is isomorphic to the group of 2 by 2 matrices with entries in $\mathbb{Z}/5\mathbb{Z}$ of the form

$$\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$$

where $a \in \mathbb{Z}/5\mathbb{Z}$ and $b \in (\mathbb{Z}/5\mathbb{Z})^*$

4. Consider the ring $R = \mathbb{Z}[x]$.

a. Show that $I = (5, x^2 - 2)$ is a prime ideal of R.

b. What is R/I?

5. Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

a. Find the minimal polynomial of M.

b. Is M diagonalizable? Explain.

6. Let p be a prime. Show that for any nonabelian group G of order p^3, the commutator subgroup of G is equal to the center of G.

7. Suppose for some group G, $|\mathrm{Aut}(G)| = 1$, show that G is abelian and every element of G has order at most 2.

8. Let p be a prime, G be a finite group of order divisible by p and P be a normal p-subgroup of G. Show that P is contained in every Sylow p-subgroup of G.

9. Let p and q be primes and G be a group of order pq^r for some $r \geq 1$. If $q \nmid p - 1$ and $p \nmid q^i - 1$ for all $1 \leq i \leq r$, show that G is nilpotent. For what r, must G be abelian?

10. Determine all the projective ideals of the ring $\mathbb{Z}/180\mathbb{Z}$. Justify your answer.