ALGEBRA QUALIFIER AUGUST 2018

Instructions: There are 10 problems below. Start each problem on a separate sheet, each labeled with your student code and the problem number. For uniformity, please only use one side of the paper when writing your solutions. If you use more than one sheet for a problem, please label accordingly.

(1) Show that any group of order $3^3 \cdot 13$ is not simple.

(2) Let N be a normal subgroup of a group G. Show that N contains the commutator subgroup if and only if G/N is abelian.

(3) Show that $\mathbb{Z}[\sqrt{-13}]$ is not a unique factorization domain.

(4) Let R be a commutative ring. If P and Q are both flat R-modules, show that $P \otimes_R Q$ is flat.

(5) Give a single representative for each similarity class of 4×4 matrices A with coefficients in $\mathbb{F}_2$ which satisfy $A^3 = I$.

(6) What is the splitting field K of $x^4 + 4$ over $\mathbb{Q}$? Find the Galois group of K over $\mathbb{Q}$.

(7) Prove that the group of non-singular $n \times n$ upper triangular matrices with coefficients in a field is solvable.

(8) Prove that the automorphism group of the algebraic closure of $\mathbb{F}_p$ is uncountable.

(9) Prove that in any commutative ring the intersection of all prime ideals is the set of all nilpotent elements.

(10) Let α and β be two complex numbers that are algebraic over $\mathbb{Q}$ of degrees m and n respectively. Assume m and n are coprime. Prove that the field $\mathbb{Q}(\alpha, \beta)$ has degree mn over $\mathbb{Q}$.