ALGEBRA QUALIFIER JANUARY 2019

Instructions: There are 10 problems below. Start each problem on a separate sheet, each labeled with your student code and the problem number. For uniformity, please only use one side of the paper when writing your solutions. If you use more than one sheet for a problem, please label accordingly.

(1) Show that if $\phi : G \to G$ given by $\phi(g) = g^2$ is an automorphism of a group G, then G must be abelian.
(2) Show that no group of order 52 is simple.
(3) Suppose G acts on both X and Y. We say that $\phi : X \to Y$ is a G-isomorphism if ϕ is a bijection and $g\phi(x) = \phi(gx)$. Let G act on X transitively. Given $x_0 \in X$. Let H be the stabilizer of x_0 in G and L be the left cosets of H. Show that there is a G-isomorphism of X with L.
(4) Let R be a domain and X and indeterminate.
  (a) Determine the units of $R[[X]]$.
  (b) Show $R[[X]]$ is a domain.
(5) Show that every prime ideal in a finite commutative ring with unity is a maximal ideal.
(6) Find all isomorphism classes of 4×4 matrices A with coefficients in $\mathbb{Q}$ satisfying $A^6 = I$.
(7) Prove that if α and β are complex numbers of coprime degrees n and m over $\mathbb{Q}$ then the extension $\mathbb{Q}(\alpha, \beta)$ has degree mn over $\mathbb{Q}$.
(8) Prove that $\mathbb{Q})(\sqrt{5}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{5})$ is not an integral domain.
(9) Prove that the $\mathbb{Z}$-module $\mathbb{Q}$ is flat but not projective.
(10) For p a prime number compute the Galois group of $x^p - p$ over $\mathbb{Q}$.