ALGEBRA QUALIFIER AUGUST 2019

Instructions: There are 10 problems below. Start each problem on a separate sheet, each labeled with your student code and the problem number. For uniformity, please only use one side of the paper when writing your solutions. If you use more than one sheet for a problem, please label accordingly.

(1) Show $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

(2) What are the two sided ideals in the ring of $n \times n$ square matrices with coefficients in a field K? Justify your answer.

(3) Let K be a field and let G be the set of all functions $f : K \to K$ that have the form $f(x) = ax + b$ for some $a, b \in K$ with a non-zero. View G as a group under composition. Prove that G is solvable. Is G nilpotent? Justify your answer.

(4) Suppose M is a maximal proper subgroup of a group G, prove either $N_G(M) = M$ or $N_G(M) = G$. Deduce that if M is a maximal subgroup of a finite group G which is not normal then the number of non-identity elements that are contained in the conjugates of M is $(|M| - 1)(G : M)$.

(5) Let p, q, r be primes with $p < q < r$. Show that a group G of order pqr has a normal Sylow subgroup of order p, q or r.

(6) Let V be a vector space over F with v and w nonzero elements of V. Prove $v \otimes w = w \otimes v$ if and only if $w = av$ for some $a \in F$.

(7) Determine all the quotients of $\mathbb{Z}/300\mathbb{Z}$ which are projective $\mathbb{Z}/300\mathbb{Z}$-modules.

(8) Let p be a prime number. Compute the Galois group of the polynomial $x^p - p$ over $\mathbb{Q}$.

(9) Prove that the automorphism group of an algebraically closed field is uncountable.

(10) Prove that the ring $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})$ is a field.