ALGEBRA QUALIFIER JANUARY 2020

Instructions: Please hand in all of the following 10 problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

(1) Let G be a group of order 402. Prove that G is not simple.
(2) Suppose H and N are subgroups of a group G with $N \lhd G$, $|H| = m < \infty$ and $(G : N) = n < \infty$ with m and n relatively prime. Prove $H \subseteq N$.
(3) What is the order of a Sylow p-subgroup in the group S_{3p} for $p > 3$. Are the Sylow p-subgroups of S_{3p} abelian groups? Justify your answers.
(4) Prove that the ring $\mathbb{Z}[\sqrt{10}]$ is not a unique factorization domain.
(5) Prove that the ideal (x, y) generated by x and y in the ring of polynomials $\mathbb{C}[x, y, z]$ is prime, is not maximal, and is not principal.
(6) Prove that there is a ring isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$.
(7) Let $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the homomorphism

$$f(x, y, z) = (3x + 5y + 7z, 8y + 9z, 100z).$$

Find the elementary divisors of $\mathrm{Coker}(f)$.
(8) Compute the Galois group of the polynomial $x^7 - 10$ over $\mathbb{Q}$.
(9) Let x be an indeterminate. Consider the field $K = \mathbb{Q}(x)$ and the subfield $F = \mathbb{Q}(\alpha)$ of K generated by

$$\alpha = x^3 + \frac{1}{x} + 1.$$

Find the degree of the extension K/F.
(10) Let A be a complex 10×10 matrix such that $A^{10000} = 0$. Prove that $A^{10} = 0$.