ALGEBRA QUALIFIER AUGUST 2020

Instructions: There are 10 problems below. Start each problem on a separate sheet, each labeled with your student code and the problem number. For uniformity, please only use one side of the paper when writing your solutions. If you use more than one sheet for a problem, please label accordingly.

(1) Show $\mathbb{Z}[\sqrt{-13}]$ is not a Euclidean domain.

(2) Consider $G = S_{28}$.
  (a) What is the order of the Sylow 7-subgroup in G?
  (b) Is the Sylow 7-subgroup abelian? Justify your answer.

(3) Let K/F be a Galois extension with degree $[K : F] = 36$.
  (a) Show that there are extensions E_1 and E_2 of F contained in K with $[E_1 : F] = 9$ and $[E_2 : F] = 4$.
  (b) Also show that there is a nontrivial extension E_3 of F properly contained in K which is Galois over F.

(4) Suppose A is a 5×5 matrix with with coefficients in $\mathbb{F}_3$ and $A^{22} = I$. How many similarity classes are there of such matrices A? Justify your answer.

(5) Let F be a field whose characteristic is not 2 and V is a finite dimensional vector space. Prove that $V \otimes_F V \cong \wedge^2 V \oplus S^2 V$.

(6) Give examples of rings $R \neq \mathbb{Z}/3\mathbb{Z}$ for which:
  (a) $\mathbb{Z}/3\mathbb{Z}$ is a projective R-module.
  (b) $\mathbb{Z}/3\mathbb{Z}$ is an R-modules but is not projective.

  Justify your answers.

(7) Find both the splitting field and the Galois group of $f(x) = x^3 + 3$ over $\mathbb{Q}$.

(8) Prove that the exact sequence $0 \to \mathbb{Z} \xrightarrow{u} \mathbb{Z} \xrightarrow{v} \mathbb{Z}/3\mathbb{Z} \to 0$, where $u(x) = 3x$, $v(y) = y + 3\mathbb{Z}$, is not split.

(9) Find a system of generators for the ideal I^3 in the ring $\mathbb{Q}[x, y]$ where $I = (x, y)$. Find the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q}[x, y]/I^3$.

(10) Prove that the group with generators a, b and relations $a^2 = b^2 = e$ is infinite.