QUALIFIER EXAM ALGEBRA JANUARY 2021

Please write your name on the exam.
Please use COMPLETE sentences.

1. Let F_2 be the free group on two generators a and b and let G be the subgroup of F_2 generated by $bab^{-1}, b^2ab^{-2}, b^3ab^{-3}$. Prove that G is isomorphic to the free group F_3 on 3 generators.
2. Prove that the group of 2×2 invertible complex matrices is not solvable.
3. Prove that for every non-cyclic subgroup G of $\mathbb{R}$ and every real numbers $a < b$ there exists $g \in G$ such that $a < g < b$.
4. Prove that if R is a commutative ring with identity then the intersection of all prime ideals consists of the set of nilpotent elements of R.
5. Let R be the ring of all continuous functions from the real closed interval $[a, b]$ to $\mathbb{R}$. Let M be a maximal ideal in R. Prove that there exists $c \in [a, b]$ such that

$$M = \{f \in R \mid f(c) = 0\}.$$

6. Prove that the $\mathbb{Z}$-module $\mathbb{C}$ is not projective.
7. Consider the rings $A = \mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})$ and $B = \mathbb{Q}(\sqrt{3}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})$. Prove that A is a field but B is not a field.
8. Prove that the Galois group of $x^5 - 7$ is solvable but not abelian.
9. Prove that the automorphism group of the algebraic closure of $\mathbb{Q}$ is uncountable.
10. Prove that if a 10×10 matrix A with complex coefficients satisfies $A^{1000} = 0$ then $A^{10} = 0$.

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