ALGEBRA QUALIFIER AUGUST 2021

(1) Show that no group of order 72 is simple.

(2) Determine the Galois group of $x^4 + 4$ over $\mathbb{Q}$.

(3) Let $\mathbb{C}[x^2, x^5]$ be the smallest subring of the polynomial ring $\mathbb{C}[x]$ that contains $\mathbb{C}$, x^2, and x^5. Show that the ring $\mathbb{C}[x^2, x^5]$ is not a UFD.

(4) Prove that the polynomial $\sum_{i=0}^{9} t^i$ is reducible in $\mathbb{Q}[t]$. Prove that the polynomial $\sum_{i=0}^{10} t^i$ is irreducible in $\mathbb{Q}[t]$.

(5) Give an example of a commutative ring R and a non-identity non-zero element $e \in R$ which satisfies $e^2 = e$. Is the R-module $R/(e)$ projective? Justify your answer.

(6) Classify up to conjugation the 5×5 matrices A with coefficients in the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ which satisfy $A^4 = A$.

(7) Give an example of a finitely generated group G and of a subgroup H of G such that H is not finitely generated.

(8) Prove that the automorphism group of the algebraic closure of a finite field is abelian.

(9) Prove that if R is an integral domain and I is a non-zero proper ideal then the exact sequence of R-modules $0 \to I \to R \to R/I \to 0$ is not split.

(10) Prove that $\mathbb{C} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$. Prove that $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$