ALGEBRA QUALIFIER JANUARY 2022

Instructions: Please hand in all of the following 10 problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

(1) Show that no group of order 132 is simple.
(2) Let p be an odd prime and G a group of order p^3. Show that $Z(G) = [G, G]$.
(3) Let G be a group with normal subgroups H and K. Show that if G/H and G/K are nilpotent, then $G/H \cap K$ is nilpotent.
(4) Let R be a commutative ring and M be an R-module.
  (a) Show that the set
  $$\mathrm{ann}_R(M) := \{r \in R \mid rm = 0 \text{ for all } m \in M\}$$
  is an ideal of R.
  (b) Let $R = k[x^2, xy, y^2]$ and $M = R/(x^2, xy) \oplus R/(xy, y^2)$. Determine $\mathrm{ann}_R(M)$.
(5) Show that the ring $\mathbb{Z}[\sqrt{10}]$ is not local and not principal.
(6) Let $R = \mathbb{Z}/12\mathbb{Z}$. Determine up to isomorphism all the quotients of R^2 which are projective R-modules.
(7) Prove that if V is a finite dimensional vector space over a field K of characteristic zero and if $u, v \in \mathrm{End}_K(V)$ then $uv - vu \neq 1$ in the ring $\mathrm{End}_K(V)$. Give an example showing that the above may fail for V of infinite dimension.
(8) Determine the splitting field E of $x^3 + 2$ over $\mathbb{Q}$. Then determine the Galois group of $E/\mathbb{Q}$.
(9) Prove that for every finite group G there is a finite Galois field extension L/K whose Galois group is isomorphic to G.
(10) Give an example of a finite field extension that has infinitely many intermediate fields. Justify your answer.