ALGEBRA QUALIFIER AUGUST 2022

Instructions: Please hand in all of the following 10 problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

(1) Let G be a group which satisfies $|\mathrm{Aut}(G)| = 1$, show that G is abelian and that every element of G is of order at most 2.

(2) Let H be the subgroup of $\mathbb{Z}^4$ generated by $(2,2,2,2)$, $(2,4,8,0)$ and $(8,8,0,4)$. Determine $\mathbb{Z}^4/H$ as a product of cyclic groups.

(3) Let $R = M_2(\mathbb{Z}/6\mathbb{Z})$ be the ring of 2×2 matrices with entries in $\mathbb{Z}/6\mathbb{Z}$. Determine all ideals of R.

(4) Determine if $\mathbb{Z}[\sqrt{-11}]$ is a UFD.

(5) Suppose F is a field whose characteristic is not 2 and V is a finitely generated F-vector space. Prove that $V \otimes_F V \cong S^2(V) \oplus \wedge^2(V)$.

(6) Let $R = \mathbb{Z}/900\mathbb{Z}$. Determine all ideals of R which are injective R-modules. Are any of these ideals flat R-modules? Justify your answer.

(7) Suppose A is a 4×4 matrix with coefficients in $\mathbb{F}_3$ satisfying $A^{20} = I$. How many similarity classes are there of such matrices A?

(8) Determine the Galois group of $x^3 - 3x + 1$.

(9) Let K be a Galois extension of F with $[K : F] = 36$.

  (a) Show there exist extensions E_1 and E_2 of F contained in K with $[K : E_1] = 4$ and $[K : E_2] = 9$.

  (b) Show there exists an extension E of F contained in K which is Galois over F.

(10) Let K be a field. Prove that the ring $K[[x]]$ of power series with coefficients in K is local and principal.

(11) Let I and J be two distinct maximal ideals in the commutative ring R. Prove that one has an exact sequence

$$0 \to IJ \to I \oplus J \to R \to 0.$$

Prove that if in addition that I and J are projective R-modules then IJ is a projective R-module.

(12) Prove that if K and L are two fields of different characteristics then there is no field homomorphism between K and L.

The students were given 12 problems accidentally. We graded all 12, but treated the exam as if it had 10 problems with extra credit.