ALGEBRA QUALIFIER JANUARY 2023

Instructions: Please hand in all of the following 10 problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

(1) Suppose N is a normal subgroup of a finite group G and
$$\gcd(|N|, (G : N)) = 1.$$
Show that N is the unique subroup of G of order $|N|$.

(2) Show that S_A acts on $P(A)$, the power set of A via $\sigma \cdot B = \{\sigma(b) \mid b \in B\}$. When $A = \{1, 2, 3, 4\}$ determine the following:
  (a) The orbit of $\{1, 2\}$.
  (b) The stabilizer of $\{1, 3, 4\}$.
  (c) The fixed set of $(12)(34)$.

(3) Prove that the power series ring $\mathbb{C}[[x]]$ is local.

(4) Show that $k[x^2, x^5]$ is not a UFD.

(5) Give an example of a non-split exact sequence of modules over the ring $\mathbb{Z}[i]$. Give an argument showing that your example is correct.

(6) Let $R = \mathbb{Z}/180\mathbb{Z}$. Determine all the projective ideals of R. Give an example of an ideal of R which is not projective.

(7) For a vector space V over a field K denote its dual, $V^* = \mathrm{Hom}_K(V, K)$. Give an example of a vector space V such that V^{**} is not isomorphic to V. Give an argument showing that your example is correct.

(8) Suppose A is a 4×4 matrix with coefficients in $\mathbb{F}_3$ satisfying $A^3 + A = 0$. Determine all isomrophism classes of A. Please list precisely one matrix for each class.

(9) Let K be a field and and let $K(t)$ be the field of rational functions in the variable t. Let
$$f = \frac{t^3 + 1}{t^2 + 1} \in K(t).$$
Find the degree of the field extension $K(t)/K(f)$.

(10) Suppose that K is a Galois extension of F of degree 72. Show that there is a field $F \subseteq E \subseteq K$ such that E is Galois over F.