## ALGEBRA QUALIFIER AUGUST 2023

Instructions: Please hand in all of the following 10 problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

(1) Suppose $H$ is a subgroup of a finite group $G$ and

$$\gcd(|H|, (G : H)) = 1.$$

Is it the case that $H$ is the unique subroup of $G$ of order $|H|$? If the answer is yes, give a proof. Otherwise give a counterexample.

(2) Suppose that $G$ is a group satisfying $|\mathrm{Aut}(G)| = 1$. Show that $G$ is abelian and that every element of $G$ has order at most 2.

(3) Prove that the group of upper triangular matrices with real coefficients is solvable.

(4) Find all the maximal ideals in the ring $\mathbb{Z}/100\mathbb{Z}$. Furthermore, if $S \subset \mathbb{Z}$ is the multiplicative set consisting of all powers of 100 find all the maximal ideals in the ring $S^{-1}\mathbb{Z}$.

(5) Let $R$ be a commutative ring and $x, y \in R$ be nilpotent elements. Prove that $x + y$ is nilpotent.

(6) Consider the product $R = \prod_p \mathbb{Z}/p\mathbb{Z}$ of all finite fields $\mathbb{Z}/p\mathbb{Z}$ where $p$ runs through the set of all prime integers and let $I = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ be the direct sum of the fields $\mathbb{Z}/p\mathbb{Z}$.
   (a) Prove that $I$ is a proper ideal of $R$.
   (b) Moreover prove that there is a ring homomorphism from $\mathbb{Q}$ to $R/I$.

(7) Let $R = \mathbb{Z}/120\mathbb{Z}$. Determine all abelian groups of order at most 200 which are injective $R$-modules.

(8) Suppose $A$ is a $5 \times 5$ matrix with coefficients in $\mathbb{F}_2$ satisfying $A^7 = I$. Determine all isomorphism classes of $A$. Please list precisely one matrix for each class.

(9) Suppose that $K$ is a Galois extension of $F$ of degree 56. Show that there is a field $F \subseteq E \subseteq K$ such that $E$ is Galois over $F$.

(10) Prove that the automorphism group of the algebraic closure of any finite field is uncountable.