ALGEBRA QUALIFIER JANUARY 2024

Instructions: There are 10 problems on this exam. For best results, we encourage you to attempt all 10 problems. Start each problem on a new page and put only your code word (not your banner ID number) on each page. Before turning the exam in, please **place the problems in order** and number all the pages. Clear and concise answers with good justification will improve your score.

(1) Show that there is no simple group of order 80.

(2) Let R be a Boolean ring i.e. $x^2 = x$ for all $x \in R$. Prove that R is commutative.

(3) Give an example of non-split exact sequence of $\mathbb{C}[x]$-modules. Justify your answer.

(4) Prove that if R is a commutative ring, $f \in R$ is an non-zero element, and S is the multiplicative system $S = \{f^n \mid n \geq 0\}$ then we have a ring isomorphism $S^{-1}R \simeq R[x]/(fx - 1)$.

(5) Prove that the ring $\mathbb{Z}[\sqrt{10}]$ is not a unique factorization domain.

(6) Let $R = \mathbb{Z}/100\mathbb{Z}$. Determine all abelian groups of order at most 125 which are projective R-modules.

(7) Prove that:
 (a) $(\mathbb{Z}/3\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) = 0$.
 (b) $(\mathbb{Z}/4\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

(8) Suppose A is a 4×4 matrix with coefficients in $\mathbb{F}_{11}$ satisfying $A^{20} = I$. Determine the **number** of isomorphism classes of A. It is not necessary to list a matrix for each class.

(9) Prove that the Galois group of the polynomial $x^5 - 2$ over $\mathbb{Q}$ is non-commutative.

(10) Give examples of fields $F \subseteq E \subseteq K$ where:
 (a) Both E and K are separable over F.
 (b) E is separable over F, but K is not separable over F. Can you say anything about the separability of K over E for this example?