Complex Analysis Qualifying Examination

January 1995

Directions :

1. **You are trying to convince the reader that you know what you are doing. To that end we suggest you be *clear, concise* and *complete.***
2. **Solve all nine problems.**
3. **Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages.**

Terminology : A *domain* is a non-empty open connected set in the complex plane.

1. (a) Find the first 3 non-vanishing terms of the Taylor series expansion of $\tan z$ around the origin.

   (b) Also find its radius of convergence.

2. Let $P_1, P_2, \cdots, P_n$ be n arbitrary points of a plane and let $\overline{PP_k}$ denote the distance between P_k and a variable point P. If P is confined to the closure of a bounded domain Ω, show that the product $\prod_{k=1}^{n} \overline{PP_k}$ attains its maximum on the boundary of Ω.

3. Evaluate
$$\int_{|z|=1} \frac{1-\cos z}{(e^z-1)\cdot \sin z} dz.$$

4. Show that $z^5+15z+1=0$ has one root in the disc $|z|<\frac{3}{2}$, four roots in the annulus $\frac{3}{2}<|z|<2$.

5. Find the fallacy in the following 'proof':

   Let m and n be two arbitrary integers. Then we have
$$e^{2m\pi i} = e^{2n\pi i}, \quad \therefore \left(e^{2m\pi i}\right)^i = \left(e^{2n\pi i}\right)^i.$$
   It follows that
$$e^{-2m\pi} = e^{-2n\pi}.$$
   Since $-2m\pi$ and $-2n\pi$ are all real, we must have
$$-2m\pi = -2n\pi, \quad \therefore m=n.$$

6. Find a conformal mapping of the domain

$$\left\{ z \in \mathbf{C};\ |z| < 1,\ \Im z > \frac{\sqrt{2}}{2} \right\}$$

onto the unit disc.

7. Construct a meromorphic function having simple poles with residue 1 at Gaussian integers $\omega_{mn} = m + in$ $(m,\ n \in \mathbf{Z})$.

8. True or false:

   (a) If $w = f(z)$ maps Ω_z conformally onto Ω_w, then $f'(z)$ never vanishes in Ω_z.

   (b) Any two annuli can be mapped conformally onto each other.

   (c) Any two non-intersecting circles can be mapped to a pair of concentric circles by a Möbius transformation.

   (d) A bounded (real-valued) harmonic function in the plane $\mathbf{C}$ is a constant.

   (e) If $\{f_k(z)\}_{k=1}^{\infty}$ is a sequence of univalent (one-to-one) analytic functions which converges uniformly on every compact subset of a domain Ω, to a non-constant function $f(z)$ in a domain Ω, then $f(z)$ is a univalent analytic function in Ω.

   (f) There is a non-constant entire function having both 2π and i as its periods; i.e.

$$f(z + 2\pi) = f(z) = f(z + i) \quad \text{for all} \quad z \in \mathbf{C}.$$

9. Prove or disprove two of the six propositions in the previous problem.