Complex Analysis Qualifying Examination

January 1997 **Name:** _____

Directions:

1. You are trying to convince the reader that you know what you are doing. To that end we suggest your presentation be *clear, consise* and *complete.*

2. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your name* in each page.

Questions:

Part A: Answer *four* out of the five questions below.

1. Find the Möbius transformation that maps the left half-plane $\{z \in C \mid \Re z < 1\}$ to the unit disc $\{w \in C \mid |w| < 1\}$ and has $z = 0$ and $z = 1$ as fixed points.

2. Prove that if $w = f(z)$ is holomorphic in the disc $D(0,2)$ then $w = f(z)$ has a Taylor expansion centered at $z_0 = 0$ which converges for $|z| \leq 1$.

3. Answer *only one* of the following questions:
(a) Evaluate the integral
$$\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}$$
(b) Evaluate the integral
$$\int_0^{\infty} \frac{x \sin x}{1+x^2} dx$$

4. Characterize all the entire functions that satisfy $f^{(n)}(0) = 1/n!$, for $n = 0, 1, 2, \ldots$.

5. Classify the singularities of the following functions in the *extended* complex plane $C \cup \{\infty\}$, and find the singular part at each of the isolated singular points:

(a) $\dfrac{1-\cos z}{z^4}$;

(b) $\sqrt{1-\sin z}$;

(c) $\dfrac{1-z^3}{1-z^2}$;

Part B: Answer *four* out of the six questions below.

1. Answer *only one* of the following questions:
(a) Characterize all the meromorphic functions $f(z)$ that have simple poles at $z = n$, $n = 1, 2, 3, \ldots$, with residue $res(f(z), z = n) = n^2$.
(b) Characterize all the entire functions $f(z)$ that have simple roots at $z = \sqrt{n}$, where $n = 1, 2, 3, \ldots$

2. Answer *only one* of the following questions:
   (a) State and prove Liouville's theorem.
   (b) State and prove the Casorati-Weierstrass theorem.

3. (a) Prove the argument principle.
   (b) How many roots does the polynomial $p(z) = z^4 + z + 1$ have in the first quadrant?

4. Find the image of the region $\{z \in C \mid 0 < \Re z < \pi,\ 0 < \Im z\}$ under the mapping $w = \cos z$.

5. If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

6. Prove the reflection principle: if $f(z)$ is a continuous function in upper half-plane $\Omega = \{z \in C \mid \Im z \geq 0\}$, analytic in the interior of Ω, and $f(z)$ takes real values on the real axis, then $f(z)$ can be extended to an entire function.