Complex Analysis Qualifying Examination

August 1997 **Name:** _____

Directions:

1. You are trying to convince the reader that you know what you are doing. To that end we suggest your presentation be *clear, consise* and *complete.*

2. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your name* in each page.

Questions:

Part A: Answer *four* out of the six questions below.

1. Find the image of the quarter disc $\Omega = \{z \in C \mid |z| \leq 1,\ \Re z \geq 0,\ \Im z \geq 0\}$ under the map $w = \frac{1}{2i}\left(z - \frac{1}{z}\right)$.

2. Assume that $w = f(z)$ is an entire function, and that a and b are two positive constants so that $f(z)$ satisfyes $|f(z)| \leq a + b|z|^2$ for all $z \in C$. Prove that $f(z)$ is a polynomial of degree no larger than two.

3. Answer *only one* of the following questions:
   (a) Evaluate the integral
   $$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx, \text{ where } 0 < \alpha < 1$$
   (b) Evaluate the integral
   $$\int_{-\infty}^\infty \frac{\cos kx}{(1+x^2)^2}\,dx, \text{ where } 0 < k < \infty$$

4. Assume that $\{f_n(z)\}_{n=1}^\infty$ is a sequence of analytic functions defined on the region Ω, such that $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on compact subsets of Ω. Show that $f(z)$ is analytic in Ω and that $\lim_{n\to\infty} f_n'(z) = f'(z)$ uniformly on compact subsets of Ω.

5. Let $u(x,y)$ be the bounded harmonic function in the upper half-plane $\{z = x + iy \in C \mid y > 0\}$ that has the boundary value
   $$u(x,0) = \operatorname{sgn} x = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$
   Find a harmonic conjugate $v(x,y)$ of $u(x,y)$.

6. Find the residue of the following functions at the indicated singularities:
   (a) $\frac{\sin^2 z}{z^5}$ at $z = 0$
   (b) $\frac{\sqrt{1-z}}{z^2}$ at $z = 0$
   (c) $\frac{1}{e^z - 1}$ at $z = 2\pi i$

Part B: Answer *four* out of the six questions below.

1. Answer *only one* of the following questions:
   (a) Find a meromorphic function $f(z)$ that has simple poles at $z = \sqrt{n}$, $n = 1, 2, 3, \ldots$, with residue $res(f(z), z = n) = 1$.
   (b) Find an entire function $f(z)$ that has double roots at $z = \sqrt{n}$, where $n = 1, 2, 3, \ldots$

2. Answer *only one* of the following questions:
   (a) State and prove Rouche's theorem.
   (b) State and prove the argument principle.

3. Show that the equation $(z-2)^2 = e^{-z}$ has two distinct roots in the disc $|z-2| \leq 1$.

4. Prove that if $f(z)$ is an entire function without any roots, then there is an entire function $g(z)$ so that $f(z) = e^{g(z)}$.

5. Answer *only one* of the following questions:
   (a) Prove that if $u(z)$ is a non-constant harmonic function in the domain Ω, then $u(z)$ does not have local maximum in Ω.
   (b) Let $u(x, y)$ be a harmonic function on the entire plane, and let $v(x, y)$ be a harmonic conjugate of $u(x, y)$. Assume that $u(x, y) \leq v^2(x, y)$ for all $(x, y) \in C$. Prove that both $u(x, y)$ and $v(x, y)$ must be constant.

6. Assume that $f(z)$ is analytic in the unit disc $D = \{z \in C \mid |z| < 1\}$, $f(0) = 0$, and $|f(z)| \leq 1$ for $|z| < 1$. Prove that $|f(z)| \leq |z|$ for $|z| < 1$.