Complex Analysis Qualifying Exam

January 2005

Directions: Do all of the following problems. Show all of your work, and justify all of your calculations.

1. Show that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

2. Classify *all* of the singularities and find the associated residues for each of the following functions:

(a) $\dfrac{\sin^2 z}{z^4}$

(b) $\dfrac{z^2 + 2z - 1}{(z-1)(z+3)^2}$.

3. Let $p(z)$ be a polynomial. Show that that there exists an infinite sequence $\{z_j\} \subset \mathbb{C}$ such that $p(z_j) = \mathrm{e}^{z_j}$ for each j.

4. Let $f : D(0,1) \mapsto D(0,1)$ be a conformal map.

(a) If $f(0) = 0$, show that $f(z) = \omega z$ for some $\omega \in \partial D(0,1)$.

(b) If $f(0) \neq 0$, show that there exists an $a \in D(0,1)$ and $\omega \in \partial D(0,1)$ such that

$$f(z) = \omega \frac{z - a}{1 - \overline{a} z}.$$

5. State some version of Rouche's theorem, and then use it to show that

$$f(z) := z\mathrm{e}^{3-z} - 1$$

has only one real zero in $D(0,1)$.

6. Consider

$$f(z) := \sum_{n=1}^{\infty} \mathrm{e}^{-n} \sin(nz).$$

It is clear that $f(0) = 0$. Determine the largest subset $S \subset \mathbb{C}$ for which $\{0\} \subset S$ and $f(z)$ is analytic for all $z \in S$.

7. Show that there exists an entire function $g(z)$ such that

$$\sin \pi z = z\mathrm{e}^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

8. Let $\gamma \subset \mathbb{C}$ be the square centered at $z = 0$ with vertices at $z = \pm 2 \pm 2\mathrm{i}$. Compute

$$\oint_\gamma \frac{z}{z^3 + 1}\, \mathrm{d}z,$$

where γ is traversed once in the counterclockwise direction.