Complex Analysis Qualifying Exam

January 2006

Directions: Do all of the following problems. Show all of your work, and justify all of your calculations.

1. Let

$$f(z) := \frac{e^z}{(z-1)^4}.$$

(a) Classify *all* of the singularities and find the associated residues.

(b) Determine the Laurent expansion of f centered at $z = 1$.

(c) If $\mathcal{C}$ denotes the positively oriented circle of radius 2 centered at $z = 0$, evaluate

$$\oint_{\mathcal{C}} f(z)\, dz.$$

2. Let

$$\Pi_u := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Find a conformal mapping $f : \Pi_u \mapsto D(0,3)$ such that $f(3+2i) = 0$.

3. Let

$$f(z) := \frac{1}{(z-1)(z-2)}.$$

Write $f(z)$ as a Laurent series centered at $z = 0$ which converges on the annulus $1 < |z| < 2$.

4. Let $f : D(0,1) \mapsto D(0,1)$ be holomorphic and satisfy $f(0) = 0$. What does Schwarz' lemma say about f? Prove it.

5. For each $n \in \mathbb{N}$ set

$$p_n(z) := \sum_{j=0}^{n} (-1)^j \frac{z^{-2j}}{j!}.$$

(a) For each fixed $n \in \mathbb{N}$, show that $p_n(z) = 0$ has precisely $2n$ solutions.

(b) For a given $\rho > 0$, show that there is an $N(\rho)$ such that if $n > N(\rho)$, then all of the zeros of $p_n(z)$ lie within $D(0,\rho)$.

6. Show that

$$\int_{-\infty}^{+\infty} \operatorname{sech}^2(x) \cos(2x)\, dx = \frac{2\pi}{\sinh(\pi)}.$$

7. Let

$$f(z) := \frac{\sin(z^{1/2})}{z^{1/2}}.$$

(a) Show that $f(z)$ is entire.

(b) Let $p_n(z)$ be a polynomial of order $n \geq 1$. For each $A \in \mathbb{C}$ show that there exist an infinite number of distinct solutions to

$$p_n(z)f(z) = A.$$

8. Consider

$$f(z) := \prod_{j=1}^{\infty} \left(1 - \frac{z}{j^3}\right).$$

(a) Show that $f(z)$ is entire.

(b) What is the order of $f(z)$?