Complex Variables Fall 2006
MS/PhD Qualifying Examination

Instruction: Complete all problems.

1a) State Rouché's theorem and use it to show that all zeros of the polynomial

$$p(z) = z^4 + 6z + 3$$

lie in the circle $|z| < 2$.

b) How many zeros of $p(z)$ lie in the annulus $1 < |z| < 2$?

2) Classify the singularities in $\mathbb{C}$ of the functions

$$f(z) = \frac{z - \sin z}{z^4}$$

and

$$g(z) = \frac{1}{z^2(z+1)} + \sin\left(\frac{1}{z}\right) .$$

3) Let

$$f(z) = \frac{1}{z^2(e^z - e^{-z})}, \quad 0 < |z| < \pi .$$

Compute the first three non–zero terms of the Laurent expansion of $f(z)$ in $0 < |z| < \pi$.

4) Let $f(z)$ and $g(z)$ be entire functions satisfying

$$|f(z)| \leq 10|g(z)| \quad \text{for all} \quad z \in \mathbb{C} .$$

Does it follow that there exists $\lambda \in \mathbb{C}$ with

$$f(z) = \lambda g(z) \quad \text{for all} \quad z \in \mathbb{C} \ ?$$

Give a proof or a counterexample.

5) Let $f(z)$ be an entire function for which the real part

$$\operatorname{Re} f(x + iy) = u(x, y)$$

is a bounded function. Does it follow that $f(z)$ is a constant function? Give a proof or a counterexample.

6) Evaluate

$$\int_0^{\pi} \frac{dt}{5 + 4\cos t} \ .$$

7) Evaluate

$$\int_0^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)} \, dx \ .$$

8) Let $a \in \mathbb{C}$, $|a| > 1$, and let $f(t)$ denote the 2π–periodic function

$$f(t) = \frac{1}{a + e^{it}}, \quad t \in \mathbb{R} \ .$$

Write $f(t)$ as a Fourier series,

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikt} \ .$$

Determine the Fourier coefficients.

Hint: Use the geometric sum formula.