

UNM Dept. of Mathematics and Statistics  
Ordinary & Partial Differential Equations  
Qualifying Examination

Fall 2010

*Instructions:* There are six (6) problems on this examination. Work all problems.

1. (15 points) Consider the following ODE system  $\begin{cases} x' = -x^2y + y - y^3 \\ y' = -x^3 + x^5 + x^3y^2 \end{cases}$ .

- (a) Find all equilibrium points.
- (b) Find linearized systems at points  $(0, 0)$ ,  $(1, 0)$  and  $(1/\sqrt{2}, 1/\sqrt{2})$ .
- (c) Determine the type and stability of each of these 3 points for the respective linearized systems.

2. (25 points) Consider the following initial-value-problem (IVP)

$$yy'' + (y')^2 + f(y) = 0, \quad y(t)|_{t=0} > 0, \quad y'(t)|_{t=0} > 0, \quad t \geq 0,$$

where  $f(y)$  is a continuous function such that  $f(y) \leq -f_0y^{2+\epsilon}$ ,  $f_0 > 0$  and  $\epsilon > 0$ , for all  $y$ . Hint: use a change of variable to reduce the order of the equation.

- (a) Prove that the solution of the IVP blows up in finite time (i.e.  $|y(t)| \rightarrow \infty$  for  $t \rightarrow T < \infty$ ).
- (b) Find estimate for the blow up time  $T$ .
- (c) Rewrite that IVP (by a change of variable) as a conservative system in a potential.

3. (15 points) Find similarity solutions to the problem

$$u_t = u_{xx}, \quad t > 0, x > 0; \quad u(x, 0) = 0, \quad x > 0; \quad u(0, t) = 1, \quad t > 0.$$

Namely, look for a solution  $U(x, t) = f(x/\sqrt{t})$  and reduce the problem to a boundary value problem for an ordinary differential equation for  $f(z)$ , where  $z$  is the similarity variable.

4. (15 points) Solve the problem

$$xu_x + u_y = 1, \quad x \in \mathbb{R}, \quad y > 0, \quad u(x, 0) = \exp(x).$$

5. (15 points) Let  $U \subset \mathbb{R}^n$  be an open set. Show that a function  $v \in C^2(U)$  that satisfies the mean-value property

$$v(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} v dS$$

for each closed ball  $B_r(x)$  of radius  $r$  centered at  $x$  with  $B_r(x) \subset U$  is necessarily harmonic. In the above,  $\omega_n = n\pi^{n/2}/\Gamma(\frac{n}{2} + 1)$  is the surface area of the unit sphere in  $\mathbb{R}^n$ .

6. (15 points) Show that the Cauchy problem for Laplace's equation is ill-posed. Consider the problem in two dimensions

$$\begin{aligned}\Delta u &= 0, & -\infty < x < \infty, & \quad y > 0, \\ u(x, 0) &= f(x), & u_y(x, 0) &= g(x), & -\infty < x < \infty\end{aligned}$$

Construct a sequence of separated solutions

$$u_n(x, y) = \frac{1}{n} Y_n(y) \cos nx$$

such that  $u_n(x, 0) \rightarrow 0$ ,  $(u_n)_y(x, 0) \rightarrow 0$  as  $n \rightarrow \infty$ , while  $u_n(x, 1) \rightarrow \infty$ . How can this be used to show that the solution does not depend continuously on the initial data?