COMPLEX ANALYSIS QUALIFYING EXAM F 2012

We, Aug. 15, 9–12
Department of Mathematics and Statistics
University of New Mexico

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) Let Γ denote the positively oriented circle of radius 3 centered at the origin and let

$$f(w) = \int_\Gamma \frac{z^2 + z - 2}{z - w} \, dz, \quad |w| \neq 3 \ .$$

   Evaluate
   a) $f(2)$;
   b) $f(-2)$;
   c) $f(4)$.

2) Find all solutions $z = x + iy$ of the equation

$$\sin z = 2$$

   and sketch the solutions z as points in the complex plane.

3) a) Sketch the region in the complex plane consisting of all z with

$$|z - 4| > |z| \ .$$

   b) Sketch the region in the complex plane consisting of all z with

$$Im(z^2) > 0 \ .$$

4) Evaluate

$$\int_0^\infty \frac{\cos x}{x^2 + 1} \, dx$$

   and justify your computation.

5) Let Γ denote the straight line from the origin to the point $\pi + 2i$. Evaluate

$$\int_\Gamma \cos(z/2) \, dz \ .$$

6) a) Find the bilinear transformation

$$w = f(z) = \frac{az + b}{cz + d}$$

   for which

$$f(0) = 0, \quad f(2) = 4, \quad f(i) = 1 - i \ .$$

   b) Find the fixed points of the transformation.
   c) What is the image of the region $\{y \geq 1\}$ under the transformation?

7) a) Show: If t is real and

$$z = \frac{1 + it}{1 - it}$$

then $|z| = 1$.

    b) Let z be any complex number with $|z| = 1, z \neq -1$. Show that one can write

$$z = \frac{1 + it}{1 - it}$$

for some real t. Is t unique?

8) Let w denote an n–th root of unity, $w \neq 1$. Evaluate

$$1 + 2w + 3w^2 + \ldots + nw^{n-1} \ .$$

(Find a simple expression for the sum which is based on the geometric sum formula.)