COMPLEX ANALYSIS QUALIFYING EXAM S 2013

Friday, Jan. 11, 9–12
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Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) a) State (a version of) Rouché's theorem.
   b) Use the theorem to prove that the polynomial

$$p(z) = 3 + az + 2z^4$$

has exactly one zero in the open unit disk if $a \in \mathbb{C}, |a| > 5$.

2) Prove the formula

$$\frac{1}{2} + \cos z + \cos(2z) + \cdots + \cos(nz) = \frac{\sin((n+\frac{1}{2})z)}{2\sin(\frac{z}{2})}$$

for all $n = 1, 2, \ldots$ and all $z \in \mathbb{C}$ with $\sin(z/2) \neq 0$.

3) Assume that the power series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

defines a holomorphic function for $|z| < 1$ and assume that

$$|a_1| > \sum_{j=2}^{\infty} j|a_j| \ .$$

Show that the function $f(z)$ is one–to–one in the open unit disk, i.e.,

$$|z_1| < 1, |z_2| < 1, z_1 \neq z_2 \quad \text{implies} \quad f(z_1) \neq f(z_2) \ .$$

Hint: Write the difference between two values of $f(z)$ as an integral of $f'(z)$.

4) Let Q denote the square

$$Q = \{z = x + iy \ : \ -1 \leq x \leq 1, \ -1 \leq y \leq 1\} \ .$$

Determine the image of Q under the exponential function,

$$exp(Q) = \{e^z \ : \ z \in Q\}$$

and sketch the image in the complex plane.

5) Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ denote an entire function satisfying the estimate

$$|f(z)| \leq Me^{|z|} \quad \text{for all} \quad z \in \mathbb{C}$$

for some constant M. Prove that the coefficients a_j satisfy

$$|a_j| \le M\Big(\frac{e}{j}\Big)^j, \quad j = 1, 2, 3, \ldots$$

6) Find Laurent expansions for

$$f(z) = \frac{4z}{(z+1)(z-3)}$$

valid in

$$\begin{array}{ll} (a) & \{z \ : \ 1 < |z| < 3\} \ ; \\ (b) & \{z \ : \ |z| > 3\} \ . \end{array}$$

7) Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} \ .$$

8) a) If

$$f(z) = \frac{1}{z^2(e^z - e^{-z})}, \quad 0 < |z| < \pi \ ,$$

find the first three nonzero terms of the Laurent expansion for $f(z)$ valid in $\{z : 0 < |z| < \pi\}$.

b) Let Γ denote the positively oriented circle of radius 3 centered at the origin. Evaluate

$$\int_\Gamma f(z)\, dz \ .$$