Complex Analysis Qualifying Exam
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Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet. Clear and concise answers with good justification will improve your score.

1. a) State a version of Rouché's theorem.
   b) Let $a \in \mathbb{C}, |a| > e$. Use Rouché's theorem to prove that the equation $e^z = az^n$ has n solutions (not necessarily distinct) in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

2. Let $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$ denote the open unit disk and let $f : D \to \mathbb{C}$ denote a holomorphic function with $|f(z)| \leq 2$ for all $z \in D$. Derive an estimate for $|f^{(3)}(i/3)|$.

3. Let $V = \{z \in \mathbb{C} \ : \ 0 < |z| < 1\}$ denote the open unit disk with $z = 0$ removed. Assume that $f : V \to \mathbb{C}$ is holomorphic and not identically zero.
   a) Is it possible that there is a sequence $z_n \in V$ with $z_n \to 0$ and $f(z_n) = 0$ for all n?
   b) Assume, in addition, that f has a pole at $z = 0$. Answer the same question.

4. Let Γ denote a smooth curve in $\mathbb{C}$ from $P = -i$ to $Q = i$.
   a) Does the integral $\int_\Gamma ze^z \, dz$ depend on the particular curve Γ or only on the endpoints of Γ?
   b) Determine all possible values of the integral.

5. Let Γ denote the curve with parametrization $z(t) = 1 + t(i - 1), \quad 0 \leq t \leq 1$.
   Compute $\int_\Gamma \sqrt{z} \, dz$ where $\sqrt{z}$ denotes the principal branch of $\sqrt{z}$ on $\mathbb{C} \setminus (-\infty, 0]$, i.e., $\sqrt{z}$ is the analytic continuation of the real function $\sqrt{x}, 0 < x < \infty$, to the domain $\mathbb{C} \setminus (-\infty, 0]$.

6. Find the Möbius transformation
   $$w = \frac{az + b}{cz + d}$$
   from the disc $D = \{z \in \mathbb{C} : |z| < 1\}$ onto the half plane $H = \{w \in \mathbb{C} : \text{Re } w < 1\}$ which sends $z = 1$ to $w = 1$ and $z = 1/2$ to $w = 0$. Then find the images of $z = 0$ and $z = \infty$.

7. a) Where is the transformation $w = \cos z$ conformal?
   b) Find the image of the region $G = \{z \in \mathbb{C} \mid 0 < \text{Re } z < \pi/4, \ 0 < \text{Im } z < \infty\}$ under the mapping $w = \cos z$.

8. Let $f(z) = \sum_{j=0}^\infty a_j z^j$ denote a holomorpic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $0 < r < 1$. The 2π–periodic function
   $$F(\theta) = f(re^{i\theta}), \quad \theta \in \mathbb{R} \ ,$$
   has the Fourier coefficients
   $$c_n = \frac{1}{2\pi} \int_0^{2\pi} F(\theta) e^{-in\theta} \, d\theta \text{ for } n = 0, \pm 1, \pm 2 \ldots$$
   How are the c_n related to the a_j? Note that the c_n are defined for *all* integers n.