COMPLEX ANALYSIS QUALIFYING EXAM
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Instructions: At the top of each page you use, write the secret code that you shared with Ana and label all problems appropriately. Complete all problems to get full credit. Start each problem on a new page, number the pages. Use only one side of each sheet. Clear and concise answers with good justification will improve your score.

1. Consider the function $f(z) = \dfrac{z+6}{z^2-2z-3}$.
   (a) Expand $f(z)$ in a Taylor series about $z = 0$ and find its radius of convergence.
   (b) Expand $f(z)$ in a Laurent series in the annulus $1 < |z| < 3$.

2. Consider the function $f(z) = u(x,y) + \mathrm{i}v(x,y)$, where
$$u(x,y) = \frac{x^3-y^3}{x^2+y^2}, \quad v(x,y) = \frac{x^3+y^3}{x^2+y^2}$$
   for $z = x+\mathrm{i}y \neq 0$ and $u(0,0) = v(0,0) = 0$. (a) Verify that u, v are continuous in a neighborhood of $z = 0$ and satisfy the Cauchy–Riemann Equations at $z = 0$. (b) Show that $f'(0)$ does not exist.

3. If $f(z)$ is entire and $f(x) = f(ix)$ for all $x \in (1,2)$, show that $f(-z) = f(z)$ for all $z \in \mathbb{C}$.

4. Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^3}$. Justify your computations.

5. Consider the function $f(z) = \dfrac{e^{1/z}}{1-2z}$. (a) Find and classify all the singularities of $f(z)$ on the Riemann sphere. (b) Does $|f(z)| \to \infty$ as $|z| \to 0$? (c) Show that $\displaystyle\oint_{|z|=1} f(z)dz = -\pi i$.

6. Suppose $f(z)$ is analytic on $\mathbb{C}\backslash\{0\}$ and $|f(z)| \leq |z|^{5/2} + \dfrac{1}{|z|^{1/2}}$ for all $z \in \mathbb{C}\backslash\{0\}$. Prove that $f(z)$ is a polynomial of degree at most 2.

7. Find how many solutions (counting multiplicity) the equation $\sin z = ez^4$ has on the unit disk $|z| < 1$. Justify your answer.

8. If $f(z)$ is analytic and $|f(z)| \leq 1$ on the unit disk $|z| < 1$, show that $|f'(0)| \leq 1 - |f(0)|^2$.