COMPLEX ANALYSIS QUALIFYING EXAM
AUGUST 12, 2022

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Instructions: At the top of each page you use, write the secret code that you shared with Ana and label all problems appropriately. Complete all problems to get full credit. Start each problem on a new page, number the pages. Use only one side of each sheet. Clear and concise answers with good justification will improve your score.

1. (a) Find a linear fractional transformation $w = f(z)$ for which
$$f(0) = 0\ , \quad f(2) = 4\ , \quad f(i) = 1 - i\ ,$$
and find also its inverse.
   (b) Clearly $z = 0$ is a fixed point. Are there any other fixed points?
   (c) Describe the image of the region $1 \leq \mathrm{Im}(z) < \infty$ in the w-plane.

2. For all complex numbers z and all positive integers n, show that $|\mathrm{Im}(z^n)| \leq n|\mathrm{Im}(z)||z|^{n-1}$.

3. (a) Find the first three nonzero terms of the Laurent expansion of $f(z) = \dfrac{1}{z^2(e^z - e^{-z})}$ in the annulus $0 < |z| < \pi$.
   (b) Evaluate $\displaystyle\oint_{|z|=3} f(z)\, dz$

4. Find all entire functions $f(z)$ that satisfy $|f(z)| \leq e^{\mathrm{Im}(z)}$ for all $z \in \mathbb{C}$.

5. Determine how many solutions $z^7 - 7z^4 + z^3 - 3z = 1$ are in the annulus $1 < |z| < 2$.

6. Consider the series $f(z) = \displaystyle\sum_{n=0}^{\infty} (n+1)(z+1)^n$.
   (a) Determine a circle, as large as possible, where the series defines a holomorphic function.
   (b) Evaluate the series and show that $f(z)$ has a holomorphic extension in $\mathbb{C} \setminus \{0\}$.

7. Let f be an analytic function on the (open) unit disk such that $|f(z)| \geq \sqrt{|z|}$ for all z on the unit disk. Show that $|f(z)| \geq 1$ on the unit disk.