ODE/PDE Qualifying Exam For Summer 2002

Instruction: Complete all four problems.

1. Consider the following initial value problem for a function $u(x,t)$,

$$u_t + (2t - x)u_x = 0, \quad u(x,0) = \cos x \ ,$$

where $x \in \mathbb{R}$ and $t \geq 0$.

a) Determine the characteristic lines in the (x,t) plane.

b) Sketch the characteristic line passing through the point $(x,t) = (0,0)$.

c) What is the value of $u(x,t)$ along the characteristic determined in b)?

d) Compute $u(x,t)$ and check your answer.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with compact support. You may use that the solution of Poisson's equation

$$-\Delta u(x) = f(x), \quad x \in \mathbb{R}^3 \ ,$$

is given by

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} \, dy \ ,$$

where

$$|z| = \sqrt{z_1^2 + z_2^2 + z_3^2}$$

denotes the Euclidean norm in $\mathbb{R}^3$.

Show that there is a constant $C > 0$, which depends on f but not on x, with

$$|u(x)| \leq \frac{C}{1+|x|}, \quad x \in \mathbb{R}^3 \ .$$

Hint: The estimate is easy for bounded x, thus the essential difficulty is to show the stated decay of $|u(x)|$ for large $|x|$.

3. Consider Newton's equation

$$x''(t) = -\frac{\partial U}{\partial x}$$

with $U(x)$ being continuously-differentiable function of x. Prove that
a) If $U(x) \geq U_0$ for some U_0, the solution of Newton's equation can be extended for all $t > 0$.

b) Find an example of $U(x)$ unbounded from below leading to singular solutions $x(t)$.

4. Consider a differential equation in $\mathbb{R}^n$:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}),$$

where $\mathbf{F}(\mathbf{x})$ is assumed Lipshitz so all the necessary theorems about existence and uniqueness of solutions apply. Let $\mathbf{x}(\mathbf{p}, t)$ be the solution of this differential equation with initial condition $\mathbf{x}(\mathbf{p}, t = 0) = \mathbf{p}$. We define the ω-limiting set as follows. A point $\mathbf{q}$ is in the ω-limiting set if there is an initial point $\mathbf{p}$ and an increasing unbounded sequence of points t_n, $n = 1, 2, \ldots$:

$$0 \leq t_1 < t_2 < \ldots < t_n < \ldots \qquad \lim_{n \to \infty} t_n = \infty$$

such that

$$\lim_{n \to \infty} \mathbf{x}(\mathbf{p}, t_n) = \mathbf{q},$$

In other words, the ω-limit set is the set of all accumulation points of $\mathbf{x}(\mathbf{p}, t)$ for $t \to \infty$.
a) Prove that the ω-limit set is invariant and closed.
b) Find the ω-limit set for the system

$$x' = y + f(r)x \qquad y' = -x + f(r)y,$$

where $r = \sqrt{x^2 + y^2}$ and $f(r) = (r - 1)(r - 2)(r - 3)$. *Hint* Use polar coordinates.