ODE/PDE qualifying exam, August 2003

Complete all five problems. Write the last four digits of your SSN (*not* your name) on each sheet.

1. The Green's function $G(r, \xi)$ for Helmholtz operator on $x \in [-\infty, \infty]$ has to satisfy

$$\frac{\partial^2}{\partial x^2} G(x, \xi) - G(x, \xi) = \delta(x - \xi)$$

   and must decay at $x = \pm\infty$. Here δ is the Dirac delta function.

   Show that $G(x, \xi) = G(\xi, x)$.

   Hint: The solutions of differential equation with zero right-hand side are $\exp(\pm x)$. Patch these solutions together, with coefficients depending on ξ, so that the resulting function is bounded and continuous in x and has a jump of 1 in its derivative at $x = \xi$.

2. Consider differential equation

$$y''(t) + 4y'(t) + 3y(t) = f(t)$$

   with $f(t)$ being a periodic function, $f(t + T) = f(t)$.
   Prove that:

   (a) There is a periodic solution $y_p(t)$ of this equation.

   (b) This periodic solution is unique.

   (c) Any solution of the initial value problem $y(t)$ converges to the periodic solution $y_p(t)$.

   Hint: Write this equation as a system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

3. In this problem, you are to perform the following tasks for the two differential equations

$$z(t)z''(t) + z'(t)^2 = 1 - z(t), \tag{1}$$

   and

$$z(t)z''(t) = 1 - z(t). \tag{2}$$

   (a) Show that $z(t) = 1$ is an equilibrium point for the each equation.

(b) Integrate each of (1, 2) once, getting an integration constant in each equation. *Hint:* Multiply (1) by $z\,z'$.

(c) In the integrated version of (1), find an exact solution satisfying the initial condition $z(0) = 0$. *Hint:* You should get a quadratic function in t.

(d) Draw a phase portrait in (z, z') plane for each equation with arbitrary integration constants close to the values $(z = 1, z' = 0)$. Show that the solutions of each equation oscillates by showing that the phase plane curve circles the critical point.

(e) Find a linearization of (1, 2) around $z(t) = 1$ by writing $z(t) = 1 + \epsilon y(t)$ and keeping only first order in ϵ terms in each equation. Solve the resulting equations for $y(t)$. Discuss the behavior of these solutions.

Note about the physical significance of (1, 2). You can ignore this digression if you are not interested in physics of these equations.
Insert an empty tube (say, a straw from a drink) into a glass full of water vertically. If we close the top of the tube with a finger when inserting it, the trapped air will not allow water in the tube to rise to the level of surrounding liquid outside the tube. If we then release the top of the tube, the inside and outside level of water will equilibrate. Eventually, the level of water inside the tube will be exactly equal to the level of water outside the tube, but how does the equilibration occur? A theory of this phenomenon was constructed by E. Loresnceau et al (*Phys Fluids*, **14** (6), 2002). The theory considers the evolution of the fluid column $z(t)$ in some conveniently re-scaled coordinates. $z = 1$ is the equilibrium, and the equation of motion for $z(t)$ is (1) for $z' > 0$ and (2) if $z' < 0$, and the prime denotes the derivative with respect to time t. The switch of the sign in the dissipation leads to two different equations for $z' > 0$ and $z' < 0$, so the full solution has to be obtained numerically.

4. This problem concerns finding the eigenvalues and eigenfunctions for the Laplace operator with Dirichlet boundary conditions on the unit square in the plane. In the plane, the Laplace operator is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,.$$

(a) Let $0 \leq x, y \leq 1$ and then find all complex valued smooth functions $u = u(x, y)$ and complex λ that are solutions of

$$\Delta u = \lambda\, u\,, \quad u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0\,.$$

(b) Enumerate all of the solutions of the previous problem in the form $\lambda_{m,n}$, $\phi_{m,n}$ where

$$\Delta \phi_{m,n} = \lambda_{m,n}\, \phi_{m,n}\,,$$

and the $\phi_{m,n}$ are orthonormal in the inner-product

$$< f, g >= \int_0^1 \int_0^1 f(x, y)\, g(x, y)\, dx\, dy\,.$$

where f and g are smooth functions on the unit square. Make a sketch of the complex plane that indicates the positions of the $\lambda_{m,n}$. Do any of the eigenvalues have multiple eigenfunctions?

(c) If f is a smooth function on $0 \leq x, y \leq 1$ then use the previous information to solve

$$\Delta u = f\,, \quad u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0\,.$$

(d) Let $L = \Delta$ be the linear operator defined on smooth functions on the unit square that are zero on the boundary of the square. Use integration by parts to show that

$$< L\, f, g >= - < \nabla f, \nabla g >\,,$$

where ∇ is the gradient. Use this result to show that the Laplacian is negative and symmetric:

$$< L\, f, g > \leq 0\,, \quad < L\, f, g >=< f, L\, g > \,.$$

What does this imply about the eigenvalues (or spectrum) of L? Do your results agree with part (b) above?

5. If $u = u(x,t)$ is a smooth function of $x \in \mathbb{R}$, $t > 0$, then the initial-value problem for the diffusion equation is

$$\frac{\partial u}{\partial t} = \Delta u \,, \quad u(x,0) = f(x)$$

where it f is a given function in $\mathbb{R}$. It may be helpful to know that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\,.$$

(a) Is the diffusion equation elliptic, parabolic or hyperbolic? State the strongest maximum/minimum principle for solutions of the initial value problem.

(b) Use the Fourier transform,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{\imath x \xi}\, dx \,, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\imath x \xi}\, d\xi \,,$$

to solve this initial value problem for smooth compactly supported f (or f that are mean-square integrable).

(c) The solution of the diffusion equation can be written in the form

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi, t)\, f(\xi)\, d\xi \,.$$

Show that

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\,.$$

Use these formulas to show that if $f \geq 0$ then $u \geq 0$ (provide some justification for your steps).

(d) Show that $G(x,t)$ is a *fundamental* solution of the IVP, that is, show that G is a solution of the differential equation and has initial value the Dirac delta function $\delta(x)$ (provide some justification for your steps).