ODE/PDE Qualifying Exams, Spring 2010

Instructions: Complete all problems. Start each problem on a new sheet of paper. Use only one side of each sheet of paper. Number the pages in the order you want them to be read. Identify yourself by writing your Banner ID # on each page.

Please do not write your name on the exam papers.

(1) Consider the autonomous ODE in $\mathbb{R}^d$, $u' = f(u)$, where f is locally Lipschitz. Let the general solution be given by $y = \phi(t, x)$, where $\phi(0, x) = x$.

a) Show that $\phi(t + s, x) = \phi(t, \phi(s, x))$, whenever these are defined. Conclude that $y = \phi(t, x)$ can be inverted to give $x = \phi(-t, y)$.

b) Consider the variational problem $w' = Df(\phi(t, x))w$ and its principle solution matrix, $\Phi(t, x)$, i.e., the solution matrix satisfying $\Phi(0, x) = I$. Show that $\Phi(t + s, x) = \Phi(t, \phi(s, x))\Phi(s, x)$, whenever these are defined. (This is a special case of the so-called cocycle identity). Conclude that $\Phi^{-1}(s, x) = \Phi(-s, \phi(s, x))$.

c) Comment on the existence, uniqueness issue for ϕ.

(2) Prove the following "equivalence" of autonomous flows and maps. Let $V \subset \mathbb{R}^d$, $\epsilon > 0$, and suppose $g : V \to \mathbb{R}^d$ is bounded and Lipschitz with Lipschitz constant $L \geq 0$. Then

(i) the map $\quad y_{n+1} = y_n + \epsilon g(y_n)$

(ii) and the flow $\quad \frac{dy}{dt} = \epsilon g(y)$

are equivalent in the sense that the solutions y_n and $y(t)$ of (i) and (ii), respectively, with common initial condition $y_0 = y(0) \in V$ satisfy the *nearness condition* $|y(n) - y_n| \leq \epsilon ||g||_V (1 + \epsilon L)^n$ provided that y_k and $y(t)$ remain in V for $0 \leq k, t \leq n$. Here $|\cdot|$ denotes a norm on $\mathbb{R}^d$ and $||g||_V$ denotes the sup of $|g(y)|$ over V.

Proceed as follows:

a) Let $E_n = |y(n) - y_n|$ and show that $E_{n+1} \leq (1 + \epsilon L)E_n + \epsilon^2 L||g||_V$, $n \geq 0$.

b) Solve the set of inequalities in a) and derive the nearness condition above.

c) Conclude that $|y(k) - y_k| \leq \epsilon C(T)$ for $0 \leq k \leq T/\epsilon$, where $C(T) = ||g||_V \exp(LT)$.

(3) Let

$$D = \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\}$$

denote the open unit disk with boundary

$$\partial D = \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\} \ .$$

Let $f : \partial D \to \mathbb{R}$ denote continuous function. Consider the Dirichlet problem: Find $u \in C^2(D) \cap C(\bar{D})$ with

$$\Delta u = 0 \quad \text{in} \quad D, \quad u = f \quad \text{on} \quad \partial D \ .$$

a) State and prove the maximum principle for the problem.
b) Prove uniqueness of a solution.
c) Sketch an existence proof.

(4) Consider the Cauchy–Riemann equations

$$u_x(x,y) = v_y(x,y), \quad v_x(x,y) = -u_y(x,y)$$

with initial condition (if we think of y as time)

$$u(x,0) = \cos(kx), \quad v(x,0) = 0 \quad \text{for} \quad x \in \mathbb{R} \ .$$

Here $k \in \mathbb{R}$ is a fixed wave number.

a) What is the solution?

b) For any fixed $y \geq 0$ let us measure the solution in supremums norm over x:

$$\|(u,v)(\cdot,y)\| := \sup_{x \in \mathbb{R}} \{|u(x,y)| + |v(x,y)|\} \ .$$

Is the Cauchy problem for the Cauchy–Riemann equations well–posed or ill–posed in this norm?

Hint: For the above initial condition, can you estimate the solution for $y > 0$ in terms of the solution at $y = 0$, with a constant independent of the wave number k?